CALIBRATING A MARKET FRACTION MODEL TO THE POWER-LAW BEHAVIOUR IN THE DAX 30

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ABSTRACT. Bounded rationality and heterogeneity have been developed in the recent asset pricing literature to explain complicated market behaviour, such as market booming and crashes, and various stylized facts in high frequency data, such as fat tails, volatility clustering and power-law behaviour in returns, which are difficult to be explained by the standard asset pricing theory based on rational expectations and representative agent paradigm. Guided by theoretical analysis, numerical simulations of those structure models have demonstrated a great success to generate and explain most of these stylized facts. However calibration of these heterogeneous agent models to the most of these stylized facts, in particular fat tail in return distribution and long-range dependence in return volatility, seems difficulty so far. This paper calibrates the simplest market fraction asset pricing model of fundamentalists and trend followers we have developed recently to the power-law behaviour of the DAX 30. With the parameter values of the calibrated model, we show that the autocorrelations (of returns, the absolute returns and the squared returns) of the market fraction model share the same pattern for the DAX 30. By conducting econometric analysis via Monte Carlo simulations, we characterize these power-law behaviors and find that estimates of the power-law decay indices, the (FI)GARCH parameters, and the tail index of the calibrated market fraction model match closely to the corresponding estimates for the DAX 30. The results strongly support the explanation power of the heterogeneous agent models.

JEL Classification: C15, D84, G12

1. INTRODUCTION

Traditional economic and finance theory is based on the assumptions of investor homogeneity and rational expectations. Since agents are rationally impounding all relevant information into their trading decisions, the movement of prices is assumed to be perfectly random and hence exhibit random walk behaviour. This view is important in empirical finance because it is the theoretical underpinning of the efficient markets hypothesis and asset pricing theories generally, including the optimal portfolio rules developed by Markowitz (1952) and Merton (1971), the static and intertemporal capital asset pricing model of Sharpe (1964), Lintner (1965), Mossin (1966) and Merton (1974) and models for the pricing of contingent claims beginning with the work of Black and Scholes (1973). The impressive statistical evidence in favour of market efficiency, documented by Fama (1976), has been taken as support for the random walk model and for a long time financial economists were contented with this view as the explanation of the time series behaviour of observed asset prices.

Empirical investigations of (high-frequency) financial time series in both equity and foreign exchange markets, however, show some common features, the so-called stylized facts. They include excess volatility (relative to the dividends and underlying cash flows), skewness, excess kurtosis, fat tails (the tails of distribution have a higher density than what is predicted by normal distribution, which conventionally was indicated by excess kurtosis), volatility clustering (high/low fluctuations are followed by high/low fluctuations), long-range dependence in volatility (often characterized by slow decay of autocorrelations of squared or absolute returns), and various power-law behaviour. We refer to Pagan (1996) for a comprehensive discussion of stylized facts characterizing financial time series and Lux (2004) for a recent survey on empirical evidence of various power laws. These facts are not entirely contradicted with the traditional economic and finance theory with representative agent and rational expectations, but the theory does not provide persuasive explanation on a large subset of these facts.

Various statistical models have been developed recently to characterize some of these facts successfully. Among the stylized facts, fat tail has been found across financial markets since the work of Mandelbrot (1963)[1]. Various statistical models have been developed to explain the power-law behavior. For instance the very popular (in standard textbooks on theoretical and empirical finance) GARCH processes, introduced in Engle (1982), model returns as a random process with a time-varying variance that shows autoregressive dependence. These models produce fat tails of the unconditional distribution and capture the short-run dynamics of volatility autocorrelations. However, the implied decay of the volatility autocorrelation is exponential rather than hyperbolic. Volatility clustering and long-range dependence (that is, insignificant autocorrelations (ACs) of raw returns and hyperbolic decline of ACs of the absolute and squared returns) have been extensively studied since the seminal paper of Ding, Engle and Granger (1993). Other popular statistical models include the ARCH-class

[1]For a recent example, see the study by LeBaron and Samanta (2005) on international equity markets

model and Markov switching model, see Bollerslev st al. (1994) and Hamilton (1994) for numerical applications. These models are quite successful in modeling some of these stylized facts, but they do not offer any economical explanation and generating mechanism of the facts.

As a result the literature witnessed increased attempts of modeling the financial markets by incorporating heterogeneous agents and bounded rationality, see recent surveys by Hommes (2006), LeBaron (2006), and Lux (2004). These models characterize the dynamics of financial asset prices resulting from the interaction of heterogeneous agents having different attitudes to risk and having different expectations about the future evolution of prices. One of the key aspects of these models is that they exhibit feedback of expectations—the agents' decisions are based upon predictions of future values of endogenous variables whose actual values are determined by equilibrium equations. In particular, Brock and Hommes (1997, 1998) proposed an *Adaptive Belief System* model of economic and financial markets. The agents adapt their beliefs over time by choosing from different predictors or expectations functions, based upon their past performance. The resulting nonlinear dynamical system is, as Brock and Hommes (1998) and Hommes (2002) show, capable of generating a wide range of complex price behavior from local stability to high order cycles and chaos. It is very interesting to find that adaptation, evolution, heterogeneity, and even learning, can be incorporated into the Brock and Hommes type of framework, see, Gaunersdorfer (2000), Hommes (2001, 2002), Chiarella and He (2001, 2002, 2003*b*), Chiarella *et al.* (2002), De Grauwe and Grimaldi (2006) and Westerhoff (2003). Moreover, recent works by Westerhoff (2004), Chiarella *et al.* (2005, 2006) and Westerhoff and Dieci (2006) show that complex price dynamics may also result within a multi-asset market framework. This broader framework also gives rise to rich and complicated dynamics and can be used to obtain a deeper understanding of market behavior. They are capable of explaining various market behaviour, such as the deviation of the market price from the fundamental price, market booming and crashes.

One of the most interesting questions for the above heterogeneous agent models (HAMs) is how well they can explain the stylized facts. In most of this literature so far, numerical simulations have been used to show that the models are able to generate some of the stylized facts of financial markets, including volatility clustering, skewness, kurtosis and fat tails, but not all. In particular, to explain the long-range dependence in asset returns within this framework seems a challenge task.

Recently, a number of power laws[2] have been found in financial series. This has spurred attempts at a theoretical explanation by using HAMs and the search for an understanding of the underlying mechanisms responsible for such power laws[3]. Multiplicative stochastic processes (with multiplicative and additive stochastic components)

[2]They include power-law distribution of large returns, hyperbolic decline of return autocorrelation function, temporal scaling of trading volume and multi-scaling of higher moments of returns.

[3]For instance, Farmer *et al.* (2004) attribute the power-law behavior in return distribution to liquidity fluctuation using market microstructure approach. We refer to Lux (2004) for a recent survey on empirical evidence, models and mechanisms of various financial power laws.

have been used to explain the power-law behavior in rational bubble models (see Kesten (1973) and Lux (2004)). However, as shown by Lux and Sornette (2002), the range of the exponent required for the rational bubble models is very different from the empirical findings. In addition, the rational bubble models share the conceptual problems of economic models with *fully* rational agents. Herding models of financial markets have been developed to incorporate herding and contagion phenomena.[4] Lux and Marchesi (1999) argue that the indeterminateness of the market fractions in a market equilibrium and the dependence of stability on the market fractions exist in a broad class of behavioral finance models, which is further supported by Giardina and Bouchaud (2003) and Lux and Schornstein (2005). With a stripped down version of an extremely parsimonious stochastic herding model with fundamentalists (who trade on observed mispricing) and noise traders (who follow the mood of the market), Alfarano *et al.* (2005) show that their herding model is able to produce relatively realistic time series for returns whose distributional and temporal characteristics are astonishingly close to the empirical findings. This is partly due to a bi-modal limiting distribution for the fraction of noise traders in the optimistic and pessimistic groups of individuals and partly due to the stochastic nature of the process leading to recurrent switches from one majority to another. A mechanism of switching between predictors and co-existing attractors is used in Gaunersdorfer and Hommes (2000) to characterize volatility clustering[5]. The highly nonlinear deterministic system may exhibit co-existence of different types of attractors and adding noise to the deterministic system may then trigger switches between low- and high-volatility phases. Their numerical simulations show quite satisfactory statistics between the simulated and actual data. However, the comparison with empirical facts is mainly based upon visual inspection, or upon a few realizations of the model without estimating the power-law indices.

In contrast to theoretical oriented models discussed above, there is also a rapidly expending literature of HAMs which are computational oriented and we refer this to a recent survey paper by LeBaron (2006). Computational HAMs are becoming increasingly important and they have been proved to be vert powerful to resembling the stylized facts, in particular, various power-law behaviours. One of the most important advantages of this approach is that many behavioural aspects at the micro level and details of the interacting agents can be aggregated at the macro level through computer simulations. However, as pointed by Hommes (2006), this approach face a problem of too many degree of freedom and too many parameters, which makes it difficult to assess the main causes of observed stylized facts. Even some of the theoretical oriented HAMs share the same problem. This problem makes the estimation and calibration of the HAMs model to financial data difficult.

There are a few recent attempts to estimate HAMs on economic or financial data. Baak (1999) and Chavas (2000) estimate a HAM on hog and beef market data and

[4]See Kirman (1991, 1993), Lux (1995, 1997, 1998), Lux and Marchesi (1999), Chen *et al.* (2001), Aoki and Yoshikawa (2002), and Alfarano *et al.* (2005).

[5]Other behavioral finance explanations for volatility clustering exist. Manzan and Westerhoff (2005) develop a model in which traders tend to over or under-react to the arrival of new information.

find evidence for heterogeneity of expectations. Winker and Gilli (2003) and Gilli and Winker (2003) estimate the exchange rate herding model of Kirman (1991) to the daily DM-US$ exchange rate and find significant switching of agents between fundamentalists and chartists. Other estimations of HAMs include Westerhoff and Reitz (2003) to exchange rate, Manzan (2003) and Boswijk et al. (2005) to yearly S&P 500 index, and Alfarano et al. (2005) to daily gold price returns, exchange rates and the DAX index. Recently, Amilon (2008) estimates two specifications of the extended Brock and Hommes switching models described in De Grauwe and Grimaldi (2003, 2006) to daily S&P 500 index by the use of efficient method of moments and maximum likelihood and compares the results to real data and more traditional econometric models. He finds that the model is able to generate some stylized facts, but fit is generally quite poor.

In terms of the comparison of the econometric characterizations between the simulation models and the actual data. Usually we do not have available a single simulation model, but a whole class, where each simulation model in the class corresponds to different parameter values, initial conditions, and so on. Preferably, we would like to be able to estimate the appropriate simulation model in this model class. Generally, however, this seems to be infeasible, due to the complexity of the microscopic simulation models, which makes verification of identification rather difficult, and thus proving consistency of estimation troublesome. Moreover, in case consistent estimation is possible, the likely heavily nonlinear relationship between observables and unknown parameters to be estimated might seriously complicate estimation. Therefore, in this paper, we only consider calibration of a model in a class of the market fraction (MF) model established in He and Li (2008, 2007), by choosing some model in the model class that minimizes a distance between particular actual data based parameters and simulation model based parameters, restricting attention to a subset of simulation models.

One of important issues in dealing with the estimation and calibration, as pointed out by Hommes (2006), is "*to understand the generating mechanism of the stylized facts, one would like to find the simplest HAM with a plausible behavioural story at the micro level, that still captures the most important stylized facts observed at the aggregate level. $\cdots$. Simple and parsimonious HAMs can thus help to discipline the wilderness of agent-based modeling*". It is this principle that guides the calibration of the MF model to the DAX 30 in this paper. The MF model is a simple stochastic asset pricing model, involving two types of traders (fundamentalists and trend followers) under a market maker scenario. He and Li (2008) explained various aspects of financial market behavior and established the connection between the stochastic model and its underlying deterministic system. Through a statistical analysis, He and Li (2008) show that convergence of market price to fundamental value, long- and short-run profitability of the two trading strategies, survivability of trend followers and various under- and over-reaction autocorrelation patterns of the stochastic model can be explained by the dynamics, including the stability and bifurcations, of the underlying deterministic system. Based on these results, He and Li (2007) studied the generating mechanism of the MF model to produce the volatility clustering and the long-range dependence of asset returns. The results show that heterogeneity, risk-adjusted trend chasing through a geometric learning process, and the interplay of a stable deterministic equilibrium and stochastic noisy processes can be the source of power-law distributed fluctuations. The power-law behaviour is further verified by econometric estimates via a Monte Carlo simulation. The analysis of generating mechanism and power-law decay estimation based on simulations in He and Li (2007) provide a promising prospective for calibration of the MF model to financial data in this paper.

In this paper, we first calibrate the MF model to the daily DAX index in terms of the power-law behavior in volatility. With the parameter values of the calibrated model, we show that the autocorrelations of returns, the absolute returns and the squared returns of the market fraction model share the same pattern for the DAX 30. By conducting econometric analysis via Monte Carlo simulations, we then characterize these power-law behaviors and find that estimates of the power-law decay indices, the (FI)GARCH parameters, and the tail index of the calibrated market fraction model match closely to the corresponding estimates for the DAX 30. Interpretation of the calibrated parameters shows a consistence with the power-law behaviour generating mechanism in He and Li (2007). The results provide a very positive evidence on the explanation power of the HAMs.

The remainder of the paper is organized as follows. Section 2 reviews the MF model. In Section 3 we first calibrate the MF model to the power-law behaviour in return volatility of the the DAX 30 stock market daily closing price index. We then estimate the power-law decay parameters of the autocorrelation of returns, the squared returns and the absolute returns, (FI)GARCH(1,1) parameters, and the power-law decay rates of the tail distribution for both the DAX 30 index and MF model-generated data. In Section 4, we present an explanation of the calibrated parameters of the MF model. Section 5 concludes.

2. THE MARKET FRACTION MODEL WITH HETEROGENEOUS EXPECTATIONS

The market fraction (MF) model considered in this section is a standard discounted value asset pricing model with heterogeneous agents. It is closely related to the framework of Brock and Hommes (1997, 1998) and Chiarella and He (2002). We outline the model and refer the readers to He and Li (2008) for full details.

Consider an economy with one risky asset, one risk free asset, and two types of traders with different beliefs or expectations on the future price of the risky asset. It is assumed that the risk free asset is perfectly elastically supplied at gross return of $R = 1 + r/K$, where r stands for a constant risk-free rate per annum and K stands for the trading frequency measured in units of a year.[6] Let P_t and D_t be the (ex dividend) price and dividend per share of the risky asset at time t, respectively. The first type of

[6]Typically, $K = 1, 12, 52$ and 250 for trading period of year, month, week and day, respectively. To calibrate the stylized facts observed from daily price movement in financial market, we select $K = 250$ in our discussion.

investors is called the fundamentalists (or informed traders) with a market fraction of population of n_1. The second type of investors is called trend followers (or uninformed traders) with a market fraction of population of n_2. Note that $n_1 + n_2 = 1$. Let $m = n_1 - n_2 \in [-1, 1]$, then $m = 1(-1)$ corresponds to the case when all the traders are the fundamentalists (trend followers).

For investor i, let $W_{i,t}$ be his/her initial wealth and $z_{i,t}$ be the number of shares of the risky asset held by the investor at t. Then the portfolio wealth of the investor i at $t+1$, $W_{i,t+1}$, is given by

$$W_{i,t+1} = RW_{i,t} + [P_{t+1} + D_{t+1} - RP_t]z_{i,t}. \tag{2.1}$$

Let $E_{h,t}$ and $V_{h,t}$ be the *beliefs* of type h ($h = 1, 2$) traders about the conditional expectation and variance of quantities at $t+1$ based on their information at time t. Denote by $R_{t+1}(= P_{t+1} + D_{t+1} - R\,P_t)$ the excess capital gain on the risky asset at $t+1$. Assume that trader of type h has a constant absolute risk aversion (CARA) utility functions with the risk aversion coefficient a_h (e.g. $U_h(W) = -\exp(-a_h W)$). By expected utility maximization, the optimal demand on the risky asset of trader of type h is given by

$$z_{h,t} = \frac{E_{h,t}(R_{t+1})}{a_h V_{h,t}(R_{t+1})}, \qquad h = 1, 2. \tag{2.2}$$

Assume zero supply of outside shares. Then, using (2.2), the population weighted aggregate excess demand $z_{e,t}$ is given by

$$z_{e,t} \equiv n_1 z_{1,t} + n_2 z_{2,t} = \frac{1+m}{2}\frac{E_{1,t}[R_{t+1}]}{a_1 V_{1,t}[R_{t+1}]} + \frac{1-m}{2}\frac{E_{2,t}[R_{t+1}]}{a_2 V_{2,t}[R_{t+1}]}. \tag{2.3}$$

We assume that the market price is determined by a market maker who cleans the market by taking a long (when $z_{e,t} < 0$) or short (when $z_{e,t} > 0$). At the end of period t, after the market maker has carried out all transactions, he or she adjusts the price for the next period in the direction of the observed excess demand with a speed of price adjustment of μ. To capture unexpected market news or the excess demand of noise traders, we introduce a noisy demand term $\tilde{\delta}_t$ which is an i.i.d. normally distributed random variable with $\tilde{\delta}_t \sim \mathcal{N}(0, \sigma_\delta^2)$. Based on these assumptions and (2.3), the market price is determined by

$$P_{t+1} = P_t + \frac{\mu}{2}\left[(1+m)\frac{E_{1,t}[R_{t+1}]}{a_1 V_{1,t}[R_{t+1}]} + (1-m)\frac{E_{2,t}[R_{t+1}]}{a_1 V_{2,t}[R_{t+1}]}\right] + \tilde{\delta}_t. \tag{2.4}$$

We now turn to discuss the beliefs of fundamentalists and trend followers. Denote by $F_t = \{P_t, P_{t-1}, \cdots; D_t, D_{t-1}, \cdots\}$ the common information set formed at time t. We assume that, apart from the common information set, the fundamentalists have *superior* information on the fundamental value, P_t^*, of the risky asset, which is assumed to follow a stationary process[7]

$$P^*_{t+1} = P^*_t \exp(-\frac{\sigma^2_\epsilon}{2} + \sigma_\epsilon \tilde{\epsilon}_t), \qquad \tilde{\epsilon}_t \sim \mathcal{N}(0,1), \qquad \sigma_\epsilon \geq 0, \qquad P^*_0 = \bar{P} > 0, \tag{2.5}$$

where $\tilde{\epsilon}_t$ is independent of the noisy demand process $\tilde{\delta}_t$. This specification ensures that neither fat tails nor volatility clustering are brought about by the exogenous fundamental price process. Hence, any non-normal pattern in risky asset return, discussed below, would be driven by the trading process itself.

The fundamentalists know of trend followers and consequently believe that the stock price may be driven away from its fundamental value. More precisely, we assume that the conditional mean and variance of the fundamental traders are, respectively

$$E_{1,t}(P_{t+1}) = P_t + \alpha(P^*_{t+1} - P_t), \qquad V_{1,t}(P_{t+1}) = \sigma^2_1, \tag{2.6}$$

where σ^2_1 stands for a constant variance of the fundamental value, and $\alpha \in [0,1]$ is the fundamentalists' speed of price adjustment toward the fundamental value. In general, the fundamental traders believe the market is efficient and prices converge to the expected fundamental value. A high (low) weight of α leads to a quick (slow) adjustment of expected prices towards the fundamental prices.

The trend followers, unlike the fundamental traders, are technical traders who believe the future price change can be predicted from various patterns or trends generated from the history of prices. The trend followers extrapolate the latest observed price change over a long-run sample mean of the history prices and to adjust their variance estimate accordingly. More precisely, their conditional mean and variance satisfy

$$E_{2,t}(P_{t+1}) = P_t + \gamma(P_t - u_t), \qquad V_{2,t}(P_{t+1}) = \sigma^2_1 + b_2 v_t, \tag{2.7}$$

where $\gamma, b_2 \geq 0$ are constants, and u_t and v_t are the sample mean and variance, respectively, which may follow some learning processes. The parameter γ measures the extrapolation rate and high (low) values of γ correspond to strong (weak) extrapolation from the trend followers. The coefficient b_2 measures the influence of the sample variance. Various learning schemes[8] can be used to estimate the sample mean u_t and variance v_t. Here we assume that

$$u_t = \delta u_{t-1} + (1-\delta)P_t, \qquad v_t = \delta v_{t-1} + \delta(1-\delta)(P_t - u_{t-1})^2, \tag{2.8}$$

where $\delta \in [0,1]$ is a constant. This is a limiting of a *geometric decay process* when the length of memory lag tends to infinity[9]. The selection of this process is two fold. First, traders tend to put a high weight on the most recent prices and less weight on the more

[7]The fundamental price process P^*_t is an approximation of continuous log-normal price process with zero drift and volatility of σ_ϵ.

[8]See for example Chiarella and He (2002, 2003*a*) and Chiarella, He and Wang (2006) for related studies.

[9]See Chiarella, He, Hung and Zhu (2006) for the proof. Basically, a geometric decay probability process $(1-\delta)\{1, \delta, \delta^2, \cdots\}$ is associated with the historical prices $\{P_t, P_{t-1}, P_{t-2}, \cdots\}$. The parameter δ measures the geometric decay rate. For $\delta = 0$, the sample mean $u_t = P_t$, which is the latest observed price, while $\delta = 0.1, 0.5, 0.95$ and 0.999 gives a half life of 0.43 day, 1 day, 2.5 weeks and 2.7 years, respectively.

remote prices when they estimate the sample mean and variance. Secondly, we believe that this geometric decay process may contribute to certain autocorrelation patterns, in particular the power-law feature observed in real financial markets. In addition, it is mathematically tractable.

To simplify the calculations, we assume that the dividend process D_t follows $D_t \sim \mathcal{N}(\bar{D}, \sigma_D^2)$, the expected long-run fundamental value $\bar{P} = \bar{D}/(R-1)$, and the unconditional variances of the price (σ_1^2) and dividend (σ_D^2) over the trading period are related[10] by $\sigma_D^2 = q\sigma_1^2$. Based on (2.6), $E_{1,t}(R_{t+1}) = \alpha(P^*_{t+1} - P_t) - (R-1)(P_t - \bar{P}), V_{1,t}(R_{t+1}) = (1+q)\sigma_1^2$ and hence the optimal demand of the fundamentalist is given by

$$z_{1,t} = \frac{1}{a_1(1+q)\sigma_1^2}[\alpha(P^*_{t+1} - P_t) - (R-1)(P_t - \bar{P})]. \tag{2.9}$$

Similarly, from (2.7), $E_{2,t}(R_{t+1}) = P_t + \gamma(P_t - u_t) + \bar{D} - RP_t = \gamma(P_t - u_t) - (R-1)(P_t - \bar{P}), V_{2,t}(R_{t+1}) = \sigma_1^2(1 + q + b\,v_t)$, where $b = b_2/\sigma_1^2$. Hence the optimal demand of the trend followers is given by

$$z_{2,t} = \frac{\gamma(P_t - u_t) - (R-1)(P_t - \bar{P})}{a_2\sigma_1^2(1 + q + b\,v_t)}. \tag{2.10}$$

Subsisting (2.9) and (2.10) into (2.4), the market price under a market maker is determined by the following 4-dimensional stochastic difference system

$$\begin{cases} P_{t+1} = P_t + \dfrac{\mu}{2}\bigg[\dfrac{1+m}{a_1(1+q)\sigma_1^2}[\alpha(P^*_{t+1} - P_t) - (R-1)(P_t - \bar{P})] \\ \qquad\qquad + (1-m)\dfrac{\gamma(P_t - u_t) - (R-1)(P_t - \bar{P})}{a_2\sigma_1^2(1 + q + b\,v_t)}\bigg] + \tilde{\delta}_t, \\ u_t = \delta u_{t-1} + (1-\delta)P_t, \\ v_t = \delta v_{t-1} + \delta(1-\delta)(P_t - u_{t-1})^2, \\ P^*_{t+1} = P^*_t \exp(-\dfrac{\sigma_\epsilon^2}{2} + \sigma_\epsilon\tilde{\epsilon}_t). \end{cases} \tag{2.11}$$

By applying the stability and bifurcation theory to the corresponding deterministic model and using Monte Carlo simulation to the stochastic model, He and Li (2008) conduct both analytical and statistical analysis for the model and find that the convergence of the market prices to their fundamental value, and various under and over-reaction autocorrelation patterns of returns can be characterized by the dynamics, including the stability and bifurcations, of the underlying deterministic system. Based on the characterizations, He and Li (2007) provide further evidence of the MF model on generating power-law behavior in volatility, showing that agent heterogeneity, risk-adjusted trend chasing through the geometric learning process, and the interplay of

[10] Let $\sigma_{\bar{P}}$ be the annual volatility of P^*_t and $\bar{D}_t = rP^*_t$ be the annual dividend. In this paper, we choose $\sigma_1^2 = \sigma_{\bar{P}}^2/K$ and $q = r^2$. In fact, the annual variance of the dividend $\bar{\sigma}_D^2 = r^2\sigma_{\bar{P}}^2$. Therefore $\sigma_D^2 = \bar{\sigma}_D^2/K = r^2\sigma_{\bar{P}}^2/K = r^2\sigma_1^2$. For our calibration in this paper, we choose $r = 5\%$ p.a, $\bar{P} = \$100$, $\sigma_{\bar{P}} = \sigma\bar{P}$, $\sigma_\epsilon = \sigma$ and $K = 250$.

noisy fundamental and demand processes and a stable deterministic equilibrium can be the source of power-law distributed fluctuations. In particular, the two noisy processes play different role; the noisy demand plays an important role in the generation of insignificant autocorrelations (ACs) on returns, while the significant decaying AC patterns of the absolute returns and squared returns are more influenced by the noisy fundamental process. These findings provide a solid foundation for the calibration of the model to financial date in terms of the power-law behavior in volatility. In the following sections, we first calibrate the MF model to characterize the power-law behaviour of the DAX 30 and estimate the decay indices and (FI)GARCH parameters for the calibrated model. We then used the calibrated parameters to explain the market behaviour and provide further supporting evidence on the generating mechanism discovered in He and Li (2007). In addition, we reveal the potential of the MF model to characterize the power-law behavior of tail distribution of asset returns.

3. Calibration of the Power-Law Behavior in the DAX 30

This section provides a calibration result of the MF model to characterize the power-law behaviour of the DAX 30. We start with a brief discussion on the stylized facts of the DAX 30, including both fat tail and power-law behaviour. We then discuss the calibration procedure, which is designed in principle to match the autocorrelation patterns in the returns, absolute and squared returns for the DAX 30, and present the calibration result. Based on the calibrated parameters for the MF model, we use Monte Carlo simulations to examine the effectiveness of the calibration. This includes to generate the autocorrelation patters, estimate the decay indices of the power-law behavior, and compare them with those of the DAX 30. In addition, we also used the calibration result to examine the power-law tail behaviour of the MF model comparing with the DAX 30. We demonstrate that the calibrated MF model generates closely the characterization of the power-law behaviour of the DAX 30 in the return autocorrelation and tails.

3.1. **Stylized Facts and Autocorrelations of Returns for the DAX 30.** The price index data for the DAX 30 comes from Datastream, which contains 8001 daily observations from 11 August, 1975 to 29 June, 2007. Use p_t to denote the price index for the DAX 30 at time t ($t = 0, ..., 8000$) and log returns r_t are defined as $r_t = \ln p_t - \ln p_{t-1}$ ($t = 1, \cdots, 8000$). Table 3.1 gives the summary statistics of r_t for the DAX 30, which shows many stylized facts in financial markets. We can see from Table 3.1 that the kurtosis for r_t is much higher than that of a normal distribution (which is 3). The kurtosis and studentized range statistics (which is the range divided by the standard deviation) show the characteristic fat-tailed behavior compared with a normal distribution. The Jarque-Bera normality test statistic is far beyond the critical value, which suggests that r_t is far from a normal distribution. Figures 3.1 (a) and (b) give the plots of p_t and r_t. It shows that the market volatility is changing over time and large absolute returns are more likely to be followed by large absolute returns than small absolute returns. This suggests that a suitable model for the data should have a time varying volatility and volatility clustering structure as suggested by the ARCH and FIGARCH models.

TABLE 3.1. Summary statistics of r_t.

mean	std.	skewness	kurtosis	min	max	stud. range	Jarque-Bera
0.00034	0.01244	-0.4765	10.436	-0.1371	0.0755	17.092	18735

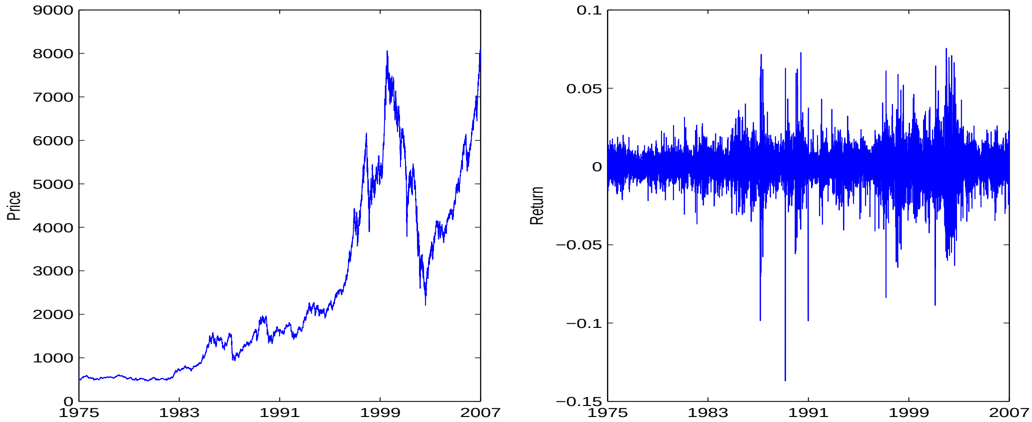


FIGURE 3.1. Time series on prices and log returns of the DAX 30 from 11 August, 1975 to 29 June, 2007.

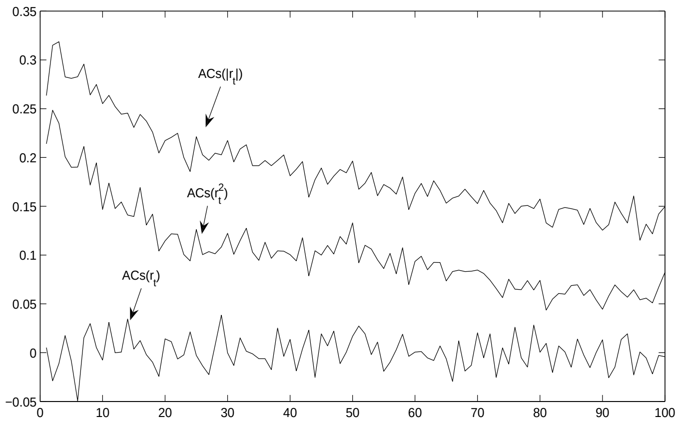


FIGURE 3.2. Autocorrelations of r_t, r_t^2 and $|r_t|$ for the DAX 30.

Apart from those reported stylized facts shared among different market indices, a well known stylized fact of stock returns is that the returns themselves contain little serial correlation, but the absolute return $|r_t|$ and the squared returns r_t^2 do have significantly positive serial correlation over long lags. For example, Ding *et al.* (1993) investigate autocorrelations (ACs) of returns (and their transformations) of the daily S&P 500 index over the period 1928 to 1991 and find that the absolute returns and the squared returns tend to have very slow decaying autocorrelations, and further, the sample autocorrelations for the absolute returns are greater than those for the squared returns at every lag up to at least 100 lags. This kind of AC feature indicates the long-range dependence or the power-law behavior in volatility. The autocorrelations for the DAX 30 are plotted in Figure 3.2, which clearly support the findings in Ding *et al.* (1993).

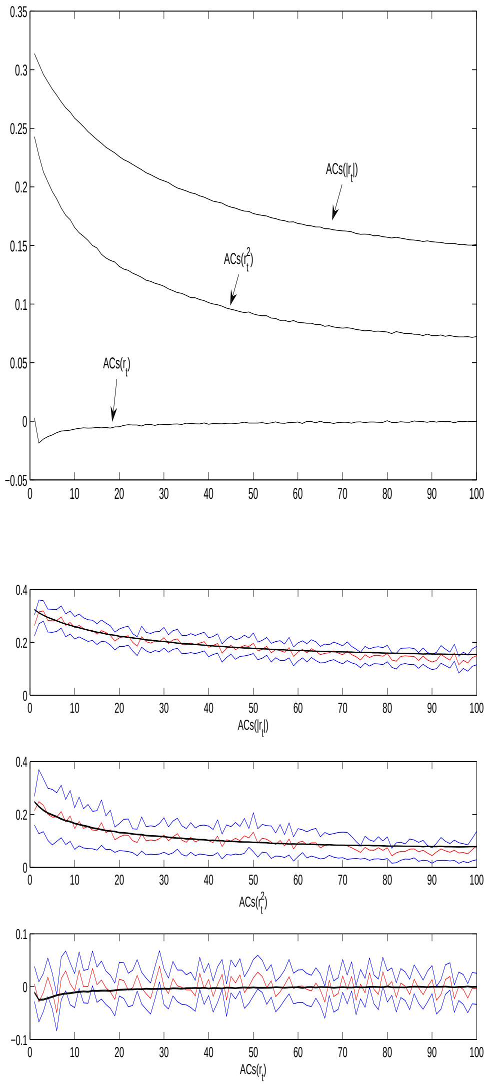
3.2. **Calibration Method and Result.** To calibrate the power-law behaviour of the DAX 30 to our MF model, in principle, we minimize the average distance between the autocorrelations of the log returns (the squared log returns, the absolute log returns) of the DAX 30 and the corresponding autocorrelations generated from the MF-models. More precisely, denote Θ the parameter space of the MF model. Let $\theta \in \Theta$ is the vector of parameters in the MF model we want to calibrate, N be the number of independent simulations of the MF model, $\widehat{\beta}_{MF}^{n}$ be the estimated autocorrelations of the n-th run of the MF model, and $\widehat{\beta}_{DAX}$ be that of the DAX 30. In calibration, we solve

$$\widehat{\theta} \in \arg\min_{\theta\in\Theta} \| \frac{1}{N} \sum_{n=1}^{N} \widehat{\beta}_{MF}^{n} - \widehat{\beta}_{DAX} \|^2,$$

for the standard Euclidian norm $\|\cdot\|$, using a generalized simplex algorithm. The parameters in the MF model are chosen to lie in the following ranges: $\alpha \in [0,1]$, $\gamma \in [0.05, 5.5]$, $a_1, a_2 \in [0.001, 6.0]$, $\mu \in [0.1, 5]$, $m \in [-1,1]$, $\delta \in [0,1]$, $b \in [0.05, 8.5]$, $\sigma_\varepsilon \in [0.005, 0.05]$, $\sigma = \sqrt{K}\sigma_\varepsilon$ and $\sigma_\delta \in [0.05, 8.5]$. However $\overline{P} = 100$ and $q = r^2 = 0.05^2$ are kept fixed. In the calibration and the subsequent econometric analysis, we ran 1,000 independent simulations over 8,000 time periods and discarded the first 1,000 time periods to wash out the possible initial noise effect. For each run of the model we obtain 8,000 observations to matche the sample size of the DAX 30. It is not possible to use autocorrelations at all lags, so we focus on a limited set of autocorrelations. In particular, we focus on lag lengths of 1 to 50, 55, 60, 65, ..., and up to 100 periods. This corresponds to 60 autocorrelations in total for return, the absolute return and squared return, respectively. Essentially, the dimension of $\widehat{\beta}_{MF}^{n}$ and $\widehat{\beta}_{DAX}$ is 180, with 60 autocorrelations estimated for each of the r_t, r_t^2 and $|r_t|$. The calibration result to the parameters of the MF model is reported in Table 3.2.

TABLE 3.2. The calibrated parameters of the MF models

α	γ	a_1	a_2	μ	m	δ	b	σ	σ_δ
0.858	8.464	6.024	0.383	0.946	-0.200	0.292	6.763	0.24	3.473



(a) (b)

FIGURE 3.3. (a) Autocorrelations of r_t, r_t^2 and $|r_t|$ for the MF model. (b) The ACs of the returns, the squared returns and the absolute returns for the calibrated MF model and the DAX 30. The smooth lines refer to the MF model while the confidence intervals are those for the DAX 30.

3.3. **The Autocorrelation Patterns of the Calibrated MF Model.** It is interesting to see whether our calibrated model is able to replicate the power-law behaviour of the DAX 30 described in Figure 3.2. Using the parameters in Table 3.2, we run 1,000 independent simulations for the MF model. For each run, we estimate the autocorrelation coefficients for returns, squared returns and absolute returns. We then take the average over the 1,000 runs and plot the ACs in Figure 3.3(a). From Figure 3.3(a), we see that for the MF model, the ACs are insignificant for the returns, but significantly positive over long lags for r_t^2 and $|r_t|$. Further, the sample autocorrelations for the absolute returns are greater than that for the squared returns at every lag up to at least 100 lags. Comparing with Figure 3.2 for the DAX 30, we see that the patterns of decay of the autocorrelation functions of return, the squared return and the absolute return are very similar. To see how well the calibrated model is able to match the autocorrelations of r_t, r_t^2 and $|r_t|$ for the DAX 30, in Figures 3.3(b), we plot the autocorrelation coefficients of returns, the squared returns and the absolute returns for the MF model together with the DAX 30 respectively. For comparison purpose, we use the Newey-West corrected standard errors and plot the corresponding confidence intervals of the ACs of the DAX 30. Figure 3.3(b) clearly indicates that all of the autocorrelations of the MF model lies inside the confidence intervals of the DAX 30.

3.4. **Estimates of Power-law Decay Index.** Besides the visual inspection of autocorrelations of r_t, r_t^2 and $|r_t|$, one can also construct models to estimate the decay rate of the autocorrelations of r_t, r_t^2 and $|r_t|$. For instance, we can semiparametrically model long memory in a covariance stationary series x_t, $t = 0, \pm 1, \ldots$, by

$$s(\omega) \approx c_1 \omega^{-2d} \qquad \text{as} \qquad \omega \longrightarrow 0^+, \tag{3.1}$$

where $0 < c_1 < \infty$, $s(\omega)$ is the spectral density of x_t, and ω is the frequency. Under (3.1), $s(\omega)$ has a pole at $\omega = 0$ for $0 < d < 1/2$ (when there is a long memory in x_t). For $d \geq 1/2$, the process is not covariance stationary. For $d = 0$, $s(\omega)$ is positive and finite. For $-1/2 < d < 0$, we have short memory, negative dependence, or antipersistence. The ACs can be described by $\rho_k \approx c_2 k^{2d-1}$, where c_2 is a constant and $\mu \equiv 2d - 1$ corresponds to the hyperbolic decay index.

Geweke and Poter-Hudak (1983), henceforth GPH, suggest a semiparametric estimator of the fractional differencing parameter d based on a regression of the ordinates of the log spectral density. Given spectral ordinates $\omega_j = 2\pi j/T \, (j = 1, 2, ..., m)$, GPH suggest to estimate d from

$$\log I(\omega_j) = c - d \log(4 \sin^2(\omega_j/2)) + v_j, \tag{3.2}$$

where v_j is assumed to be *i.i.d.* with zero mean and variance $\pi^2/6$. If the number of ordinates m is chosen such that $m = g(T)$ satisfying $\lim_{T\to\infty} g(T) = \infty$, $\lim_{T\to\infty} g(T)/T = 0$ and $\lim_{T\to\infty}(\log(T)^2)/g(T) = 0$, then the OLS estimator of d based on (3.2) has the limiting distribution

$$\sqrt{m}(\hat{d}_{GPH} - d) \xrightarrow{d} \mathcal{N}(0, \frac{\pi^2}{24}). \tag{3.3}$$

Robinson (1995) provides a formal proof for $-1/2 < d < 1/2$, Velasco (1999) proves the consistency of $\hat{d}_{GPH}$ in the case $1/2 \leq d < 1$ and its asymptotic normality in the case $1/2 \leq d < 3/4$. It is clear from this result that the GPH estimator is inconsistent with $\sqrt{T}$ convergence and in fact converges at a slower rate.

Another most often used estimator of d is developed by Robinson and Henry (1999), henceforth RH. They suggest a semiparametric Gaussian estimate of the memory parameter d, by considering

$$\hat{d}_{RH} = \arg\min_d R(d), \qquad R(d) = \log\left\{\frac{1}{m}\sum_{j=1}^{m}\omega_j^{2d}I(\omega_j)\right\} - 2\frac{d}{m}\sum_{j=1}^{m}\log\omega_j, \quad (3.4)$$

in which $m \in (0, [T/2])$. They prove that, under some conditions (see Robinson and Henry (1999)),

$$\sqrt{m}(\hat{d}_{RH} - d) \xrightarrow{d} \mathcal{N}(0, \frac{1}{4}) \quad (3.5)$$

when $m < [T/2]$ such that $1/m + m/T \to 0$ as $T \to \infty$.

A major issue in the application of the GPH and the RH estimators is the choice of m, due to the fact that there is a limited knowledge available concerning this issue, see Geweke (1998) for instance. Hence it is a wise precaution to report the estimated results for a range of bandwidths. In our study, for both the GPH and the RH estimates of d, we report the corresponding estimates for $m = 50, 100, 150, 200$ and 250, respectively. For instance, for the DAX 300, Table A.1 in Appendix A reports the GPH and the RH estimates of d for returns, the squared returns, and the absolute returns, respectively. In each panel in Table A.1, the first row reports the results from the GPH and the RH estimates with $m = 50$, the second row reports the results of the GPH and the RH estimates with $m = 100$, and so on. Table A.2 in Appendix A is arranged similarly.

For the DAX 30, we see from Table A.1 that all of the estimated d for the returns are not significant at all conventional significance levels while those for the squared returns, and the absolute returns are significant. Thus the DAX 30 displays a clear evidence of power-law for the squared and the absolute returns where d is positive, and the persistence in the absolute returns is much stronger than that in the squared returns. These results coincide with the well-established findings in the empirical finance literature.

For the calibrated MF model, the estimates of the decay rate d are reported in Table A.2 in Appendix A, where the column 'Sig%' indicates the percentage of simulations for which the corresponding estimates are significant at the 5% level over 1,000 independent simulations. We find that on average the estimates of d are insignificant for returns, but significantly positive for the squared returns and the absolute returns. This verifies that there is a clear evidence of power-law for the squared returns and the absolute returns, and also the patterns of the estimates of d for the squared returns and the absolute returns are comparable to those of the DAX 30 in Table A.1.

The above analysis has clearly demonstrated that our calibration is very effective in matching the autocorrelation patterns of the DEX 30. In the following discussion, we want to see if the calibration MF model can be used to characterize the volatility clustering and power-law tail behaviour, for which our calibration procedure is not designed.

3.5. **Volatility Clustering, Power-law and (FI)GARCH Estimates.** Another striking feature of the return series in market indices is the *volatility clustering*. A number of econometric models of changing conditional variance have been developed to test and measure volatility clustering. The most widely used one is the one introduced by Engle (1982) and its generalization, the GARCH model, introduced by Bollerslev (1986). Following their specification, for instance, if we model the returns as an AR(1) process, then a GARCH(p, q) model is defined by:

$$\begin{cases} r_t = a + br_{t-1} + \varepsilon_t, & \varepsilon_t = \sigma_t z_t, \\ \sigma_t^2 = \alpha_0 + \alpha(L)\varepsilon_t^2 + \beta(L)\sigma_t^2, & z_t \sim N(0,1), \end{cases} \tag{3.6}$$

where L is the lag operator, $\alpha(L) = \sum_{i=1}^{q} \alpha_i L^i$ and $\beta(L) = \sum_{j=1}^{p} \beta_i L^j$. Defining $v_t = \varepsilon_t^2 - \sigma_t^2$, the process can be rewritten as an ARMA(s, p) process

$$[1 - \alpha(L) - \beta(L)]\varepsilon_t^2 = \alpha_0 + [1 - \beta(L)]v_t \tag{3.7}$$

with $s = \max\{p, q\}$. Table 3.3 reports the estimates of the GARCH $(1, 1)$ model for the DAX 30, where the mean process follows an AR(1) structure. Based on the estimates, one can see that a small influence of the most recent innovation (small α_1) is accompanied by a strong persistence of the variance coefficient (large β_1). It is also interesting to observe that the sum of the coefficients $\alpha_1 + \beta_1$ is close to one, which indicates that the process is close to an integrated GARCH (IGARCH) process. Such parameter estimates are rather common when considering returns from high frequency daily financial data of both share and foreign exchange markets (see, Pagan (1996)). The GARCH implies that shocks to the conditional variance decay exponentially. However the IGARCH implies that the shocks to the conditional variance persist indefinitely.

TABLE 3.3. GARCH $(1, 1)$ Estimates for the DAX 30

$a \times 10^3$	b	$\alpha_0 \times 10^4$	α_1	β_1
0.4827	0.0539	0.0218	0.1056	0.8831
(0.1136)	(0.0127)	(0.0073)	(0.0232)	(0.0216)

Note: The numbers in parentheses are standard errors.

In response to the finding that most of the financial time series are long memory volatility process, Baillie *et al.* (1996) consider the Fractional Integrated GARCH (FIGARCH) process, where a shock to the conditional variance dies out at a slow hyperbolic rate. Later on, Chung (1999) suggests a slightly different parameterization of the model:

$$\phi(L)(1 - L)^d(\varepsilon_t^2 - \sigma^2) = \alpha_0 + [1 - \beta(L)]v_t, \tag{3.8}$$

TABLE 3.4. FIGARCH $(1, d, 1)$ Estimates for the DAX 30

a	b	$\alpha_0 \times 10^4$	d	ϕ_1	β
-0.0019	0.0012	0.0699	0.3259	0.2286	0.7716
(0.0003)	(0.0092)	(0.0248)	(0.0078)	(0.0148)	(0.0034)

Note: The numbers in parentheses are standard errors.

where $\phi(L) = 1 - \sum_{i=1}^{q} \phi_i L^i$, $\alpha_0 = \phi(L)(1-L)^d\sigma^2$, and σ^2 is the unconditional variance of the corresponding GARCH model. Table 3.4 reports the estimates of the FIGARCH $(1, d, 1)$ model for the DAX 30, where the mean process follows an AR(1) model. The estimate for the fractional differencing parameter $\hat{d}$ is statistically very different from both zero and one. This is consistent with the well known findings that the shocks to the conditional variance dies out at a slow hyperbolic rate.

TABLE 3.5. GARCH $(1, 1)$ Estimates for the Calibrated MF Model

$a \times 10^3$	b	$\alpha_0 \times 10^4$	α_1	β
-0.0836	0.0241	0.3385	0.1009	0.9050
(0.6939)	(0.0122)	(0.1040)	(0.0093)	(0.0081)
1.0	53.2	85.9	99.9	100

Note: The numbers in parentheses are the standard errors, and the numbers in the last row are the percentages that the test statistics are significant at 5% level over 1000 independent simulations. This also holds for Table 3.6.

TABLE 3.6. FIGARCH $(1, d, 1)$ Estimates for the Calibrated MF Model

a	b	$\alpha_0 \times 10^4$	d	ϕ_1	β
-0.0381	0.0264	0.11162	0.4332	0.1922	0.7490
(0.1037)	(0.0973)	(0.2275)	(0.0379)	(0.0494)	(0.0313)
74.4	66.3	4.3	89.2	89.2	95.2

For the same specifications of the GARCH and FIGARCH models, we report resulting estimates for the calibrated MF model in Tables 3.5 and 3.6, respectively. Again, all these estimates are the average of the estimations for each independent run of the calibrated model. The results from the GARCH model are astonishingly similar to that from the DAX 30, that is, a small influence of the most recent innovation is accompanied by strong persistence of the variance coefficient and the sum of the coefficients $\alpha_1 + \beta_1$ is close to one. For the estimates of the FIGARCH$(1, d, 1)$, we see that the estimate of d for the calibrated MF model is significantly different from zero and one.

3.6. **Power-law Tail Behavior.** Since the work of Mandelbrot (1963), power-law tail behavior was found in a wide range of financial time series, it has become one of the salient features in financial markets. In general, if f_{normal} is the probability density function of a normal distribution with mean μ and variance σ^2, then we have $\log f_{normal}(x) \sim -\frac{1}{2\sigma^2}x^2$ as $x \to \pm\infty$. A random variable X is said to follow a power-law or Pareto distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$ if $Pr[X > x] = (x/\beta)^{-\alpha}$, for $x \geq \beta$. In this case, $\log f_{Pareto}(x) \sim -(\alpha+1)\log(x)$ as $x \to +\infty$. Hence the difference of the tail behavior between the normal and Pareto distribution is significant.

The estimations of tail index have been studied in great detail in the Extreme Value Theory. More precisely, let $X_1, X_2, ..., X_n$ be a sequence of observations from some distribution function F, with its order statistics $X_{1,n} \leq X_{2,n} \leq ... \leq X_{n,n}$. As an analogue to the Central Limit Theorem, we know that, on average, if the maximum $X_{n,n}$, suitably centered and scaled, converges to a non-degenerate random variable, then there exist two sequences $\{a_n\}$ $(a_n > 0)$ and $\{b_n\}$ such that

$$\lim_{n\to\infty} \text{Pr}\left(\frac{X_{n,n} - b_n}{a_n} \leq x\right) = G_\gamma(x), \tag{3.9}$$

where

$$G_\gamma(x) := \exp(-(1+\gamma x)^{-1/\gamma})$$

for some $\gamma \in R$ and x such that $1+\gamma x > 0$. Note that for $\gamma = 0$, $-(1+\gamma x)^{-1/\gamma} = e^{-x}$. If (3.9) holds, then we say that F is in the max-domain of attraction of G_γ and γ is called the extreme value index. In Pareto distribution, the tail index $\gamma := 1/\alpha$ measures the thickness of the tail distribution, the bigger the γ, the heavier the tail. The estimation of γ has been thoroughly studied, see Beirlant *et al.* (2004) for a detailed account. We outline three major estimators, the Hill estimator, the Pickands estimator, and the moment estimator in Dekkers *et al.* (1989). The Hill index is defined by

$$H_{k,n} = \frac{1}{k}\sum_{j=1}^{k} \log X_{n-j+1,n} - \log X_{n-k,n}.$$

This estimator is consistent for $k \to \infty$, $k/n \to 0$ as $n \to \infty$, and under extra conditions, $\sqrt{k}(H_{k,n} - \gamma)$ is asymptotically normal with mean 0 and variance γ^2. The Pickands (1975) estimator is defined as

$$\hat{\gamma}_{P,k} = \frac{1}{\log 2}\log\left(\frac{X_{n-\lceil k/4\rceil+1,n} - X_{n-\lceil k/2\rceil+1,n}}{X_{n-\lceil k/2\rceil+1,n} - X_{n-k+1,n}}\right).$$

The simplicity of the Pickands estimator is appealing but offset by large asymptotic variance, equal to $\gamma^2(2^{2\gamma+1}+1)\{(2^\gamma - 1)\log 2\}^{-2}$. Dekkers *et al.* (1989) introduce a moment estimator, which is a direct extension of Hill index,

$$M_{k,n} = H_{k,n} + 1 - \frac{1}{2}\left(1 - \frac{H_{k,n}^2}{H_{k,n}^{(2)}}\right)^{-1},$$

where

$$H_{k,n}^{(2)} = \frac{1}{k}\sum_{j=1}^{k}(\log X_{n-j+1,n} - \log X_{n-k,n})^2.$$

They also prove the consistency and asymptotic normality.

The Hill index relies on the average distance between extreme observations and the tail cutoff point to extrapolate the behavior of the tails into the broader part of distribution. In practice, the behavior of the Hill index depends heavily on the choice of cutoff point k. This choice involves a tradeoff between bias and variance, which is well known in the non-parametric econometrics. If k is chosen conservatively with few order statistics in the tail, then the tail estimate will be sensitive to outliers in the distribution and have a high variance. On the other hand if the tail includes observations in the central part of the distribution, the variance is reduced but the estimate is biased upward. In the top panel of Figurer 3.4, we plot the Hill index over a range of tail sizes. We see that for the negative tail, the Hill index of the MF model fits in the confidence intervals of the DAX 30; for the positive tail, it fits well when k is chosen less than 500. The Pickands estimates, plotted in the middle panel of Figurer 3.4, show a larger variability. It seems that on average the estimates from the MF model with the calibrated parameters are not far away from those of the DAX 30. The moment estimates, plotted in the bottom panel of Figurer 3.4 for the MF model are smaller than those for the DAX 30 but still fit in the confidence intervals. To conclude, the MF model exhibits power-law tail behavior which is very close to that of the DAX 30.

The overall analysis in this section shows that the calibration method is very effective. The calibrated MF model is able to characterize successfully not only the power-law behaviour in autocorrelation, but also the volatility clustering and power-law tail behaviour in the DAX 30 as well.

4. Implication and Explanation of the Calibration Result

In this section, we use the calibrated parameters and the findings in He and Li (2007, 2008) to interpret the market behaviour and provide some implications and explanations on the generating mechanism of the power-law behavior of the MF model.

Based on the calibrated parameters in Table 3.2, the MF model of the DAX 30 market has the following features. The parameter value of $m = -0.2$ indicates that both the fundamentalists and trend followers are active in the market but the market is dominated by the trend followers with a majority of 60%. The higher a_1 and lower a_2 imply that the fundamentalists are more risk averse comparing to the trend followers. Among the 40% of the fundamentalists, a high value of $\alpha = 0.858$ indicates that their speed of price adjustment toward the fundamental value is high. A higher value of $\gamma = 8.464$ indicates that the trend followers extrapolate the price trend, measured by the difference between the current price and the geometric moving average of the history prices, strongly. The geometric decay rate is measured by δ and a value of $\delta = 0.292$ gives a half life of 0.56 day, implying a very quick decaying weight. The parameter $b_2 = b\sigma_1^2$ measures the influence of the sample variance v_t, in addition to the

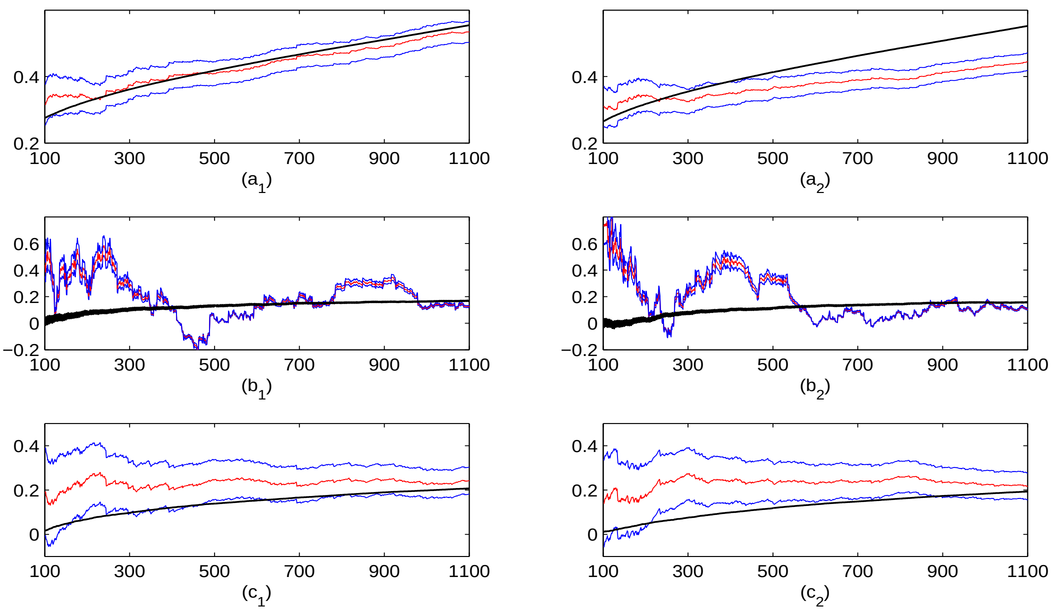


FIGURE 3.4. The tail index plots $(k, H_{k,n})$, $(k, \hat{\gamma}_{P,k})$, and $(k, M_{k,n})$ of the negative tails (a_1), (b_1), (c_1) and the positive tails (a_2), (b_2), (c_2) for the MF model and the DAX 30, respectively. The smooth lines refer to the MF model while the confidence intervals are those for the actual data.

common belief on the price volatility σ_1^2, to the estimated price volatility for the trend followers. The calibrated value of $b = 6.763$ implies that the trend followers are cautious when estimating the price volatility, though they are less risk averse (measured by the CARA coefficient a_2). The calibrated annual return volatility of $\sigma = 24\%$ is closer to the annual return volatility of $\sqrt{250} \times 0.01244 = 19.67\%$ for the DAX 30. A value of $\mu = 0.946$ indicates that the speed of the market price adjustment from the market maker is lower. The market price noise is about 3% of the average market price level. Intuitively, we have a very interesting market. The market is dominated by the trend followers. This dominance make the market less stable. However the market is balanced somehow by the activities of other market participants. More precisely, the trend followers destabilize the market in general. They are less risk averse, extrapolate the price trend strongly. However, they are bounded rational in a sense that they are cautious about their estimating price volatility when they extrapolate and follow the price trend. The fundamentalists stabilize the market price to the fundamental value in general. They are more risk averse and adjust the market price to the fundamental value quickly. A low speed of price adjustment from the market maker can reduce the market impact of the trend followers when the market becomes unstable. Therefore, due to the trend following activity, the trend followers make the market price to deviate from the fundamental value, but their cautiousness to the high volatility due to they own extrapolation, together with the activity of the fundamentalists, makes the market price move back to the fundamental value. However, due to the strong activity of the fundamentalists, the market price can move back to the fundamental price very quickly, this trend in turn is extrapolated by the trend followers, pushing the market price to the other side of the fundamental value. This price process repeats again and again.

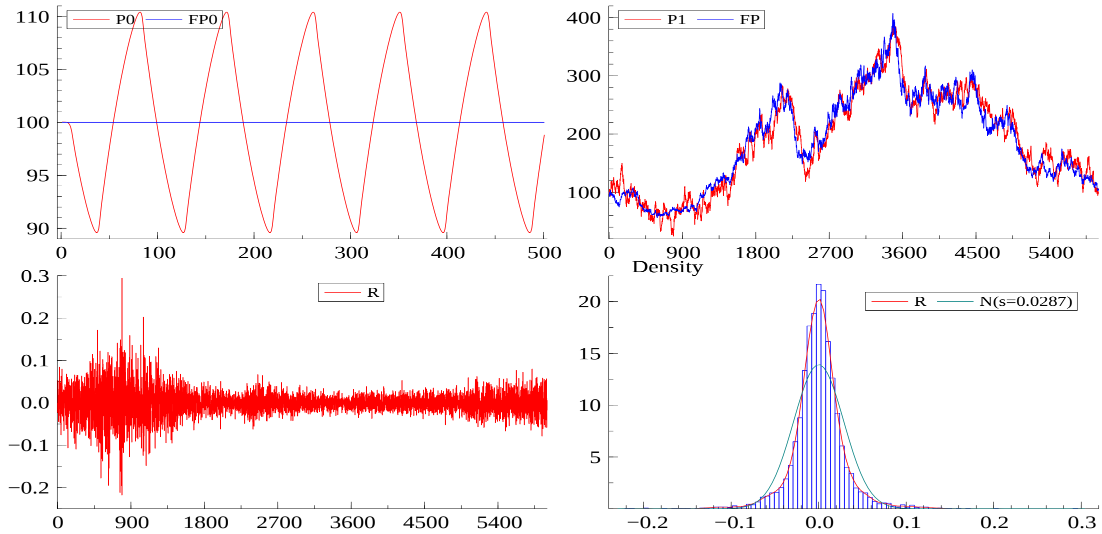


FIGURE 4.1. Price $(P0, P)$ and fundamental price $(FP0, FP)$ for the underlying deterministic model (upper left) and the stochastic model (upper right); the return (R) series (bottom left) and distribution (bottom right) of the calibrated MF model.

The above intuitive explanation based on the calibrated result of the MF model can be verified by the dynamics of the underlying deterministic model, which has been examined extensively in He and Li (2008). In fact, for the corresponding deterministic model with the calibrated parameters, the constant fundamental equilibrium becomes unstable through a Hopf bifurcation, leading to (a)periodical oscillation of the market price around the fundamental equilibrium. This is illustrated by the upper left panel in Figure 4.1. Such periodical deviations of the market prices from the fundamental value in the deterministic model are inherited by the stochastic model. The upper right panel in Figure 4.1 plots both the market price and fundamental price of the stochastic model. It shows that the market prices deviate from the fundamental prices from time to time, but follow the fundamental prices in general. In addition, the returns of the stochastic model display the stylized facts of volatility clustering (the bottom left in Figure 4.1) and non-normality in distribution (the bottom right in Figure 4.1).

One of the important contributions of this paper is that the calibration result provides a strong support on the power-law behaviour mechanism examined in He and Li (2007). In He and Li (2007), the MF model is used to examine potential source of agent-based model with heterogeneous belief in generating power-law behaviour in return autocorrelation patterns. By examining the dynamics of the underlying deterministic model and simulating the impact of two different forms of noisy processes on the deterministic dynamics, He and Li (2007) find that the interaction of fundamentalists and risk-adjusted trend chasing from the trend followers and the interplay of noisy fundamental and demand processes and the underlying deterministic dynamics can be the source of power-law behaviour. In particular, it demonstrates that, for the MF model with a chosen set of parameters near the Hopf bifurcation value of underlying deterministic model, adding noisy demand plays an important role in the generation of insignificant autocorrelations (ACs) on the returns, while the significant decaying AC patterns of the absolute returns and squared returns are more influenced by the noisy fundamental process. This potential source of power-law generating mechanism obtained in He and Li (2007) through experiment is verified from the calibration we conduct in this paper. In particular, the calibrated parameters correspond to Hopf-bifurcation induced (a)periodic oscillation of the deterministic dynamics. The same mechanism is also used in Chiarella, He and Hommes (2006). Intuitively, the calibration conducted in this paper should fit the data better than than the experiment conducted in He and Li (2007) and this intuition is confirmed by the following discussion.

To see how well the MF model is able to describe these characteristics in the DAX 30, we construct confidence intervals for the estimates based upon the DAX 30 to see if the estimates based upon the calibrated MF model lie in these intervals or not. In the following, we focus on the average estimates of the MF model rather than their accuracy since, by running the MF model independently many times, the estimates converge much faster than those of the DAX 30. Apart from checking the confidence intervals, we also construct Wald test for this purpose. For instance, for the decay index d of the returns, the squared returns or the absolute returns, we want to test whether the values of the parameter d estimated from both the DAX 30 and the MF model are the same. In other words, we want to test hypothesis

$$H_0 : d_{DAX} = d_{MF}.$$

Using the Wald test, this null hypothesis can be tested by assuming that both the number of simulations and the number of time periods for each simulation go to infinity. In the construction of the Wald test, the test statistic is given by

$$W = (\hat{d}_{DAX} - \hat{d}_{MF})\hat{\Sigma}^{-1}(\hat{d}_{DAX} - \hat{d}_{MF}),$$

where $\hat{\Sigma}$ is simply the variance of $\hat{d}_{DAX}$. The resulting test statistics are summarized in Table 4.1. In the column 'r_t', the first sub-row reports the test statistics corresponding to $\hat{d}_{GPH}$, and the second sub-row corresponding to $\hat{d}_{RH}$, and so on. Notice that the critical values of the Wald test at 5% and 1% significant levels are 3.842 and 6.635, respectively. For the returns, we see that the estimated d of the DAX 30 and the MF model are significantly different. However, for the squared returns and the absolute returns, the differences between the estimated d of the DAX 30 and the MF model are not statistically significant. The same Wald test is also conducted for the simulation experiment in He and Li (2007) and the test statistics indicate that the estimated decay parameters from the calibration is much closer to the estimates from the DAX 30, comparing to the the simulation experiment in He and Li (2007). For a more general discussion on a comparison of the simulation models with the real world data, see Li *et al.* (2006*a*, 2006*b*).

TABLE 4.1. The Wald test of d with $m = 50, 100, 150, 200, 250$

m	50	100	150	200	250
r_t	18.58	45.71	64.69	71.22	84.57
	40.80	93.16	132.1	128.6	146.3
r_t^2	0.044	1.263	0.303	0.015	0.055
	0.055	1.245	0.028	0.590	0.167
$\|r_t\|$	0.078	1.097	1.668	0.484	0.266
	0.038	0.335	0.075	0.047	0.029

The above analysis indicates that the calibration of the simple market fraction model is able to replicate most stylized facts and power-law behaviour in the DAX 30, though formal statistical tests find that not all of the estimates from the calibrated MF model could easily completely match those of the DAX 30 simultaneously (this is probably due to the simplicity of the MF model). It is this simplicity that makes it possible to identify potential sources and mechanisms and to calibrate those characteristics in the DAX 30.

5. CONCLUSION

To characterize stylized facts and power-law behaviour in financial market, we have evidenced a growing literature in HAMs to incorporate agents' heterogeneity and bounded rationality. The theoretical oriented HAMs have provided many insights into many market behaviour such as market booming and crashing, existence of multiple market equilibrium, the short-run deviation of the market price from the fundamental price and long-run convergence of the market price to the fundamental price. Combined with numerical simulations, the HAMs have been proved to be able to reproduce some stylized fact, such as non-normality in return and volatility clustering. More recently development in the HAMs has stimulated many interests in the generation mechanism of those stylized facts and in particular, the power-law behaviour. However, estimation and calibration of the HAMs to the power-law behaviour of financial data is a difficult and challenge task. This is due to the *wilderness* of the HAMs that have too many parameters to be estimated, leading to less clear understanding of mechanisms that generate the stylized facts. This is also due to our limited understanding of the interplay between the deterministic dynamics and exogenous noise processes. In addition, most of the HAMs deal with the dynamics at price level, while most of the stylized facts, in particular the power-law behaviour, are at the return level. This mismatching between theoretical models and empirical facts makes it difficult to understand the generating mechanism of the stylized facts of the HAMs in general.

To understand these issues and to overcome the difficult, He and Li (2008) develop a simple and parsimonious stochastic market fraction (MF) asset pricing model of two types of traders (fundamentalists and trend followers) under a market maker scenario. They seek to explain aspects of financial market behavior, such as market dominance, convergence of the market price to the fundamental price, and under- and over-reaction, and to characterize various statistical properties, including the convergence of the limiting distribution and autocorrelation structure, of the stochastic model by using the dynamics of the underlying deterministic system, traders' heterogeneous behavior and market fractions. A statistical analysis based on Monte Carlo simulations shows that the long-run behavior, convergence of the market prices to the fundamental price, limiting distributions, and various under and over-reaction autocorrelation patterns of returns can be characterized by the stability and bifurcations of the underlying deterministic system. The analysis underpins the mechanisms on various market behaviors (such as under/over-reactions), market dominance and stylized facts in high frequency financial markets. Based on these results, He and Li (2007) use the MF model to show that agent heterogeneity, risk-adjusted trend chasing through the geometric learning process, and the interplay of noisy fundamental and demand processes and the underlying deterministic dynamics can be the source of power-law distributed fluctuations. Those analysis of the simple and parsimonious MF model provides a foundation for our calibration in this paper.

Motivated by the mechanism investigation on the power-law properties of the simple and parsimonious MF model, this paper calibrates the model to the power-law behaviour in the DAX 30. The calibration method is based on minimization of the average distance between the autocorrelations (ACs) of the returns, the squared returns and the absoluted returns of the DAX 30 and the corresponding ACs generated from the MF model. With the parameter values of the calibrated model, we show that the ACs of the market fraction model share the same ACs pattern for the DAX 30. To characterize the power-law behaviors statistically, we conduct econometric analysis via Monte Carlo simulations and estimate the decay indices, the (FI)GARCH parameters, Hill index and related tests. We find that the calibrated MF model matches closely to the corresponding estimates for the DAX 30. We also demonstrate that the calibration parameters are consistent with the economic intuition of the model and supports the generating mechanism of the power-law behaviour of the MF model found in He and Li (2007). As a by-products, the calibrated model also generates non-normality return distribution, volatility clustering, and fat tails. Therefore the calibrated MF model can fit the most of the stylized facts observed in the DAX 30. Our results thus provide a strongly support on the explanation power of the HAMs.

It is worth emphasizing that all these interesting qualitative and quantitative features arise from the simple model with fixed market fractions. It may be interesting to extend our analysis and calibration to the model established recently by Dieci *et al.* (2006), in which part of the market fractions are governed by market mood and the rest follow some adaptive switching process. One way to start might be to estimate the model first, and then implement misspecification tests. Econometric methods, such as efficient methods of moments could be used. Allowing for market mood and switching mechanisms and using these econometric estimation approaches, we may gain a better characterization and understanding of the mechanisms driving financial markets and calibrate the model to better fit the financial data.

APPENDIX A. ESTIMATION RESULTS

TABLE A.1. The estimates of d for the DAX 30 with $m = 50, 100, 150, 200, 250$

	$\hat{d}_{GPH}$	t	p-value	95% CI	$\hat{d}_{RH}$	t	p-value	95% CI
r_t	0.0014	0.014	0.989	[-0.2005, 0.2034]	-0.0179	-0.253	0.801	[-0.1565, 0.1207]
	0.0407	0.587	0.557	[-0.0954, 0.1769]	0.0615	1.229	0.219	[-0.0365, 0.1595]
	0.0548	0.985	0.325	[-0.0542, 0.1638]	0.0829	2.031	0.042	[0.0029, 0.1629]
	0.0406	0.852	0.394	[-0.0528, 0.1340]	0.0482	1.362	0.173	[-0.0211, 0.1175]
	0.0543	1.283	0.199	[-0.0286, 0.1372]	0.0571	1.807	0.071	[-0.0048, 0.1191]
r_t^2	0.4111	3.990	0.000	[0.2091, 0.6130]	0.3785	5.353	0.000	[0.2399, 0.5171]
	0.4527	6.518	0.000	[0.3165, 0.5888]	0.4365	8.731	0.000	[0.3385, 0.5345]
	0.4053	7.288	0.000	[0.2963, 0.5143]	0.3735	9.149	0.000	[0.2935, 0.4535]
	0.3666	7.696	0.000	[0.2733, 0.4600]	0.3508	9.923	0.000	[0.2816, 0.4201]
	0.3785	8.946	0.000	[0.2956, 0.4614]	0.3605	11.40	0.000	[0.2985, 0.4225]
$\|r_t\|$	0.5242	5.087	0.000	[0.3222, 0.7261]	0.4801	6.790	0.000	[0.3415, 0.6187]
	0.5495	7.911	0.000	[0.4133, 0.6856]	0.5167	10.33	0.000	[0.4187, 0.6147]
	0.5442	9.785	0.000	[0.4352, 0.6532]	0.4914	12.04	0.000	[0.4114, 0.5714]
	0.4993	10.48	0.000	[0.4059, 0.5927]	0.4818	13.63	0.000	[0.4125, 0.5511]
	0.4797	11.34	0.000	[0.3968, 0.5626]	0.4708	14.89	0.000	[0.4088, 0.5327]

TABLE A.2. The estimates of d for the MF model with $m = 50, 100, 150, 200, 250$

	$\hat{d}_{GPH}$	t	p-value	95% CI	Sig%	$\hat{d}_{RH}$	t	p-value	95% CI	Sig%
r_t	-0.4426	-4.296	0.058	[-0.4490, -0.4362]	84.2	-0.4337	-6.134	0.039	[-0.4381, -0.4294]	91.0
	-0.4292	-6.179	0.034	[-0.4335, -0.4249]	91.9	-0.4211	-8.421	0.025	[-0.4242, -0.4180]	94.5
	-0.3924	-7.055	0.028	[-0.3958, -0.3889]	92.9	-0.3860	-9.454	0.019	[-0.3885, -0.3834]	95.4
	-0.3611	-7.580	0.023	[-0.3641, -0.3582]	94.2	-0.3533	-9.992	0.014	[-0.3555, -0.3511]	96.2
	-0.3347	-7.910	0.022	[-0.3373, -0.3320]	94.2	-0.3251	-10.28	0.015	[-0.3271, -0.3232]	96.0
r_t^2	0.3896	3.781	0.020	[0.3832, 0.3960]	91.4	0.3950	5.587	0.002	[0.3906, 0.3994]	99.1
	0.3746	5.394	0.001	[0.3703, 0.3789]	99.3	0.3807	7.615	0.000	[0.3776, 0.3838]	100
	0.3747	6.738	0.000	[0.3713, 0.3782]	100	0.3803	9.315	0.000	[0.3778, 0.3828]	100
	0.3725	7.819	0.000	[0.3695, 0.3754]	100	0.3780	10.69	0.000	[0.3758, 0.3802]	100
	0.3686	8.711	0.000	[0.3659, 0.3712]	100	0.3734	11.81	0.000	[0.3715, 0.3754]	100
$\|r_t\|$	0.4954	4.808	0.002	[0.4890, 0.5018]	98.7	0.4938	6.983	0.000	[0.4894, 0.4981]	99.8
	0.4767	6.863	0.000	[0.4724, 0.4810]	100	0.4758	9.516	0.000	[0.4727, 0.4789]	100
	0.4724	8.494	0.000	[0.4690, 0.4759]	100	0.4720	11.56	0.000	[0.4695, 0.4746]	100
	0.4662	9.787	0.000	[0.4633, 0.4692]	100	0.4665	13.19	0.000	[0.4643, 0.4687]	100
	0.4579	10.82	0.000	[0.4553, 0.4606]	100	0.4587	14.51	0.000	[0.4567, 0.4606]	100

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