Distribution and Tails of the Smith and Farmer's Model

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Abstract. We describe the distribution of the model of a limit order market by Smith et al. [2003]. We analytically prove that the tails of the price increments are fat with the tail exponent one if the initial order books are empty and that they are thin if the limit orders are collected for some (arbitrarily short) time and an initial call auction is held before the start of the trading. Hence, our results point out to the stabilizing role of the initial call auction.

Keywords: limit order market, continuous double auction, distribution, fat tails

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AMS classification: 91B26, 91B70

1. Introduction

In our paper, we deal with the recent zero intelligence model of a limit order market by Smith et al. [2003]. In Šmíd [2008], the distribution of a general model whose special case the Smith and Farmer's model is, is rigorously described; however, since the general model is quite abstract, the application of its formulas to particular models is not straightforward and deserves some mathematical work.

In the present paper, we do this job for the Smith & Farmer's model: we recursively describe the distribution of the S. & F.'s model.

Further, we pay an attention to the tails of the price increments in the model - we prove them to be fat given the empty initial order books but thin given that an initial order book is infinite and dense enough which happens if the orders are collected for some time instead of the continuous time trading and a subsequent call auction is held.

2. The S.& F. Model

By their model, S.& F. describe a limit order market[1] with discrete equidistant (log)prices and unit order sizes. The sell and buy market orders arrive with the

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[1]For a description of the functioning of limit order markets see Smith et al. [2003] or Šmíd [2008]

same fixed rate $\eta > 0$, the buy/sell limit orders with a specified price (less/greater than the best quote of the opposite type) arrive with the common fixed rate $\varsigma > 0$ per tick and the rate of cancelation of each waiting limit is equal to a fixed value $\iota > 0$. For details, see Smith et al. [2003].

3. The Distribution

For each $p \in \mathbb{Z}$ and $t \geq 0$, denote $A_t(p)$ and $B_t(p)$ the number of sell limit orders, buy limit orders respectively, with the limit price $p \in \mathbb{Z}$ waiting at $t \geq 0$. As demonstrated by Šmíd [2008] (see Example 3 therein), the random element

$$\Xi_t = (A_t(\bullet), B_t(\bullet)), \qquad t \geq 0,$$

is a (infinitely dimensional) Markov Process.

Denote

$$a_t = \min\{p : A_t(p) > 0\}, \qquad b_t = \max\{p : B_t(p) > 0\}$$

the best ask, bid respectively, and

$$\xi_t = (a_t, A_t(a_t), b_t, B_t(b_t))$$

the process of the best quotes and volumes offered at the best quotes. Demote $0 = \tau_0 < \tau_1 < \tau_2 < \dots$ the jump times of ξ.

Along with Šmíd [2008], we describe the conditional distribution of Ξ_T given Ξ_0 and the history of ξ up to T, denoted by $\bar{\xi}_T$, first.

Proposition 1. *Let T be deterministic or equal to the time of the i-th jump of ξ for some i. Then*

$$A_T(p)|\bar{\xi}_T, \Xi_0 \sim \text{Poisson}\left(\frac{\varsigma}{\iota}(1 - e^{-\iota(T-\theta_p)})\right) \circ \text{Binomial}\left(\sigma_p, e^{-\iota(T-\theta_p)}\right)$$

where $\theta_p = 0 \vee \max\{t \leq T : a_t \geq p\}$, and $\sigma_p = A_{\theta_p^-}(p)$, a symmetric formula holds for

$$B_T(p)|\bar{\xi}_T, \Xi_0$$

and

$$\dots, A_T(-1), B_T(-1), A_T(0), B_T(0), A_T(1), \dots$$

are conditionally independent given $(\bar{\xi}_T, \Xi_0)$.

Proof. The assertion is an immediate consequence of Šmíd [2008], Proposition 2

Before going on, let us introduce the variables $\zeta_1, \zeta_2, \dots,$

$$\zeta_i \in \{\mathfrak{a}, \mathfrak{b}, \mathfrak{a}^\star, \mathfrak{b}^\star, \mathfrak{a}^\times, \mathfrak{b}^\times\}, \qquad i \in \mathbb{N},$$

coding the type of the event happening at the time τ_i. Here, $\mathfrak{a}$ and $\mathfrak{b}$ mean an arrival of a buy limit order, sell limit order respectively, with limit price π such that $b_{t_{i-1}} \leq \pi \leq a_{t_{i-1}}$, further, $\mathfrak{a}^\star$ and $\mathfrak{b}^\star$ stands for a sell, buy respectively, market order arrival and, finally, $\mathfrak{a}^\times$ and $\mathfrak{b}^\times$ denote the cancelation of the best ask, bid respectively.

Proposition 2. *Denote* $s_t = a_t - b_t + 1$. *Then*

(i) $\Delta\tau_i|\bar{\xi}_{\tau_{i-1}}, \Xi_0 \sim \mathrm{Exp}\,(\mu)$ *where* $\mu = 2(\eta + \iota + \varsigma s_{\tau_{i-1}})$

(ii)

$$\mathbb{P}(\chi_i = c|\Delta\tau_i, \bar{\xi}_{\tau_{i-1}}, \Xi_0) = \mu^{-1} \times \begin{cases} \eta & \text{if } c = \mathfrak{a}^\star \text{ or } c = \mathfrak{b}^\star \\ \iota & \text{if } c = \mathfrak{a}^\times \text{ or } c = \mathfrak{b}^\times \\ \varsigma s_{\tau_{i-1}} & \text{if } c = \mathfrak{a} \text{ or } c = \mathfrak{b} \end{cases}$$

(iii) The conditional distribution of $a_{\tau_i}|\chi_i, \Delta\tau_i, \bar{\xi}_{\tau_{i-1}}, \Xi_0$ *is*

- *Dirac, concentrated in* $a_{\tau_{i-1}}$ *if* $\chi_i \in \{\mathfrak{a}^\star, \mathfrak{b}^\times, \mathfrak{b}\}$
- *uniform on* $\{b_{\tau_{i-1}}, b_{\tau_{i-1}+1}, a_{\tau_{i-1}}\}$ *if* $\chi_i = \mathfrak{a}$
- *such that, for any any* $\sigma(\bar{\xi}_{\tau_{i-1}}, \Xi_0)$*-measurable variable* p*, it holds that*

$$\mathbb{P}(a_T > p|\chi_i, \Delta\tau_i, \bar{\xi}_{\tau_{i-1}}, \Xi_0) = q_p$$

$$q_p = \prod_{p=a_{\tau_i}}^{p} [1 - e^{-\iota(\tau_i - \theta'_p)}]^{\sigma'_p} \exp\left\{-\frac{\varsigma}{\iota}\left(1 - e^{-\iota(\tau_i - \theta'_p)}\right)\right\},$$

$$\theta'_p = 0 \vee \max\{t < \tau_i : a_t \geq p\}, \qquad \sigma'_p = A_{(\theta'_p)-}(p)$$

if $\chi_i \in \{\mathfrak{b}^\star, \mathfrak{a}^\times\}$

Proof. The assertion may be proved analogously to Proposition 3 of Šmíd [2008].

4. Initial Call Auction

Suppose now that, before the beginning of the trading (e.g. over the night), the orders arrive to the market but no trading is done and the orders accumulate and, at the end of the accumulation period (time zero), the call auction is held (i.e. all the trades are made for a price maximizing the traded volume). The (limit) order flow is assumed to be the same as in the continuous model S. & F., i.e. with unit intensity ς and starting from a (fixed) minimal possible sell limit price $a_\star$, maximal buy limit price $b^\star$ respectively. For simplicity, we assume that no market orders come and no cancelations take place during the accumulation period (our results would be preserved up to constants after the inclusion of market orders and cancelations).

Under our assumptions, best ask a the time zero is

$$a_0 = \min\left\{p \in \mathbb{Z} : \sum_{\pi=a_\star}^{p} S(\pi) > \sum_{\pi=p}^{b^\star} D(\pi)\right\}$$

where $D(p)$ ($S(p)$) are the numbers of buy (sell) limit orders with limit price p having arrived until the time zero (analogous formula holds for b_0)

Given our assumptions

$$D(b^\star), S(a_\star), D(b^\star - 1), S(a_\star + 1), \ldots$$

are clearly i.i.d. Poisson with intensity $\lambda = \varsigma\theta$ where $\theta \geq 0$ is the length of the accumulation period. Even if the joint distribution of (B_0, A_0) could be quite easily determined, we will be interested only in the distribution of $(A_0(a_0+1), A_0(a_0+2), \dots)$ in the sequel: It may be easily shown that

$$A_0(a_0+1), A_0(a_0+2), \dots$$

is a sequence of i.i.d. Poisson variables with intensity λ. Quite naturally, we shall assume the order flow at the initial period to be independent of the flow of the orders starting from time zero.

In case that $\theta = 0$, i.e. there is no initial auction, we shall assume the initial order books to consist each of a single (deterministic startup) order, naturally denoted by a_0, b_0 respectively.

4.1. Tails of the i-th jump

In the present subsection, we shall deal with the right tail of a at the time of the i-th jump of ξ.

First, let us note that each jump *up* of a has to happen at one of the times

$$t_1, t_2, \dots$$

where, for each $i \in \mathbb{N}$, t_i is either an arrival of a buy market order or a cancelation of a single (sell) order with limit price $a_{t_i^-}$ - the order whose cancelation "causes" t_i is chosen according to some rule at the time of the previous jump of a (it will be suitable for us to chose the one with the highest limit price in the underlying model, see Sec. 3 of Šmíd [2008]) - clearly, $\Delta t_1, \Delta t_2, \dots$ are i.i.d. exponential with intensity $\eta + \iota$. Note also that each t_i, $i \in \mathbb{N}$, causes a jump of ξ.

It follows from the definition of the dynamics of Ξ that

$$a_{t_i} \leq d_i, \qquad i \in \mathbb{N}, \tag{1}$$

where $d_0 = a_0$ and, for each $i \in \mathbb{N}$,

$$d_i = \min\{p \in \mathbb{Z}, p > \bar{a}^i : A_{t_i^-}(p) > 0\}, \qquad \bar{a}^i = \max\{a_0, a_{t_1}, \dots, a_{t_{i-1}}\}.$$

Let $i \in \mathbb{N}$. Denote $\Theta_i = (a_0, t_1, a_{t_1}, \dots, t_{i-1}, a_{t_{i-1}}, t_i)$. It follows from Šmíd [2008], Proposition 6, that

$$A_{t_i^-}(\bar{a}^i + p)|\Theta_i \sim \text{Poisson}\left(\frac{\varsigma}{\iota} + e^{-\iota t_i}\left[\lambda - \frac{\varsigma}{\iota}\right]\right), \qquad p > 0,$$

and that

$$A_{t_i^-}(\bar{a}^i + 1), A_{t_i^-}(\bar{a}^i + 2), \dots$$

are conditionally independent given Θ_i for each $i \in \mathbb{N}$. Therefore,

$$\mathbb{P}(\Delta d_i > p|\Theta_i) = \exp\left\{-\left(\frac{\varsigma}{\iota} + e^{-\iota t_i}\left[\lambda - \frac{\varsigma}{\iota}\right]\right)p\right\}. \qquad p \in \mathbb{N}_0.$$

Since $\mathbb{P}(\Delta d_i \in \bullet | \Theta_i)$ does not depend on $a_0, a_{t_1}, \ldots, a_{t_{i-1}}$ it coincides with $\mathbb{P}(\Delta d_i \in \bullet | t_1, \ldots, t_i)$. Using it,

$$\mathbb{P}(\Delta d_i > p) = \int_0^\infty \cdots \int_0^\infty \mathbb{P}(\Delta d_i > p | s_1, \ldots, s_i) d\mathbb{P}_{t_1}(s_1) \ldots d\mathbb{P}_{t_i}(s_i)$$

$$= \int\limits_0^\infty \cdots \int\limits_0^\infty \exp\left\{ -\left(\frac{\varsigma}{\iota} + e^{-\iota \sum_{j=1}^i s_j} \left[\lambda - \frac{\varsigma}{\iota} \right] \right) p \right\} \prod_{j=1}^i \left[(\iota + \eta) e^{-(\iota+\eta)s_j} \right] ds_1 \ldots ds_i$$

$$\overset{s_j = -\frac{\ln u_i}{\iota}}{=} \left(1 + \frac{\eta}{\iota}\right)^i \exp\left\{ -\frac{\varsigma}{\iota} p \right\} \int\limits_0^1 \cdots \int\limits_0^1 \exp\left\{ \left[\frac{\varsigma}{\iota} - \lambda \right] p \prod_{j=1}^i u_j \right\} \prod_{j=1}^i u_j^{\varsigma/\iota} du_1 \ldots du_i$$

$$= \left(1 + \frac{\eta}{\iota}\right)^i \exp\left\{ -\frac{\varsigma}{\iota} p \right\} \sum_{\nu=0}^\infty \frac{\left(\left[\frac{\varsigma}{\iota} - \lambda \right] p \right)^\nu}{(\nu + 1 + \varsigma/\iota)^i \nu!}$$

(we got the last "=" by integrating the Taylor expansion of the integrand at the previous term). Since, for any $1 \le k \le i$,

$$l_k \le \frac{\nu + k}{\nu + 1 + \varsigma/\iota} \le h_k, \qquad l_k = 1 \wedge \frac{k}{1 - \varsigma/\iota} \qquad h_k = 1 \vee \frac{k}{1 - \varsigma/\iota},$$ [2]

we have that

$$\sum_{\nu=0}^\infty \frac{\left(\left[\frac{\varsigma}{\iota} - \lambda \right] p \right)^\nu}{(\nu + 1 + \varsigma/\iota)^i \nu!} \begin{Bmatrix} \ge & L_i \\ \le & H_i \end{Bmatrix} \cdot \sum_{\nu=0}^\infty \frac{\left(\left[\frac{\varsigma}{\iota} - \lambda \right] p \right)^\nu}{(\nu + i)!}$$

$$= \begin{Bmatrix} L_i \\ H_i \end{Bmatrix} \cdot \frac{1}{\left[\frac{\varsigma}{\iota} - \lambda \right]^i p^i} \left(\exp\left\{ \left[\frac{\varsigma}{\iota} - \lambda \right] p \right\} - \sum_{k=0}^{i-1} \frac{1}{k! \left[\frac{\varsigma}{\iota} - \lambda \right]^k p^k} \right)$$

where

$$L_i = [\min\{l_1, l_2, \ldots, l_i\}]^i, \qquad H_i = [\max\{h_1, h_2, \ldots, h_i\}]^i$$

(we have used the formula for the Taylor expansion of the exponential at the last "=") i.e.

$$\mathbb{P}(\Delta d_i > p) \begin{Bmatrix} \ge & L_i \\ \le & H_i \end{Bmatrix} \cdot \left(\exp\{-\lambda p\} \frac{c_i}{p^i} - \exp\left\{ -\frac{\varsigma}{\iota} p \right\} \sum_{k=0}^{i-1} \frac{c_k}{p^k} \right),$$

for some positive $c_0, \ldots, c_i$, immediately implying

$$0 < \lim_{p \to \infty} \frac{\mathbb{P}(\Delta d_i > p)}{\exp\{\lambda p\} p^i} < \infty.$$

Further, using (1) and the fact that

$$\mathbb{P}(d_i > p) \le \mathbb{P}(\Delta d_1 > p/i \text{ or } \Delta d_2 > p/i \text{ or } \ldots \text{or } \Delta d_i > p/i\ ) \le \sum_{j=1}^i \mathbb{P}(\Delta d_j > p),$$

[2] Indeed, if $1 - \varsigma/\iota \le k$ then $1 \le \frac{\nu+k}{\nu+1+\varsigma/\iota} = 1 + \frac{k-(1+\varsigma/\iota)}{\nu+1+\varsigma/\iota} \le 1 + \frac{k-(1+\varsigma/\iota)}{k} = \frac{k}{1+\varsigma/\iota}$, similarly for $1 - \varsigma/\iota > k$.

we are getting

$$\lim_{p\to\infty} \frac{\mathbb{P}(a_{t_i} - a_0 > p)}{\exp\left\{\frac{\lambda}{i}p\right\} p} < \infty$$

i.e. the tail exponent of a_{t_i} is ∞ if $\lambda > 0$ or at least one if $\lambda = 0$.

We show that the exponent is exactly one if $\lambda = 0$: Let $p > 0$ be a constant divisible by 2^i. Denote ϑ the minimum of

- the time of the first sell market order arrival
- the time of the cancelation of the initial sell order
- the time of the arrival of the first sell limit order with limit price $b_0 < \pi \leq a_0$.

It is clear that $\vartheta_1 > t_1$ implies $a_1 = d_1$. Further, consider the random event

$$E = [\vartheta > t_i, \Delta d_1 > 2p, \Delta\tau_2' + \cdots + \Delta\tau_i' < \Delta t_2, \sigma > \Delta t_2] \wedge E_2 \wedge \cdots \wedge E_i$$
$$E_\nu = [\pi_\nu \in (2^{-(\nu-1)}p, 2^{-\nu}p], \zeta_\nu = \mathfrak{b}], \qquad 2 \leq \nu \leq i,$$

where

$\tau_1' = t_1$,

τ_ν' is the time of the arrival of the first (buy or sell) limit order with relative limit price belonging to $\{1, 2, \ldots, 2p\}$

π_ν is the relative limit price of the order having arrived at τ_ν'

ζ_ν is $\mathfrak{b}$ iff the order having arrived at τ_ν' is the buy one

σ is the lifetime of the order having arrived at τ_2'.

It may be shown that $a_i - a_0 > p$ given E. Using this, we are getting

$$\mathbb{P}(a_i - a_0 > p) \geq \mathbb{P}(E) \geq P(F_1 \wedge F_2)$$

$$F_1 = [t_1 < T, \Delta d_1 > 2p],$$

$$F_2 = [\vartheta > T + \Delta t_2 + \cdots + \Delta t_i, \Delta\tau_2' + \cdots + \Delta\tau_i' < \Delta t_2, \sigma > \Delta t_2] \wedge E_2 \wedge \cdots \wedge E_i$$

where T is arbitrarily chosen constant. Since, for any deterministic T,

$$\begin{aligned}
\mathbb{P}(\Delta d_1 > p, t_1 \leq T) &= \left(\frac{\kappa}{\iota}\exp\left\{\frac{\varsigma}{\iota}p\right\}\right)\int_{e^{-\iota t}}^{1} \exp\left\{\frac{\varsigma}{\iota}pu\right\} u^{\varsigma/\iota} du \\
&= \left(\frac{\kappa}{\iota}\exp\left\{\frac{\varsigma}{\iota}p\right\}\right)\sum_{\nu=0}^{\infty} \frac{\left(\frac{\varsigma}{\iota}p\right)^\nu}{(\nu + 1 + \varsigma/\iota)\nu!}\left(1 - \exp\{-(\iota\nu + \iota + \varsigma)T\}\right) \\
&= \left(\frac{\kappa}{\iota}\exp\left\{\frac{\varsigma}{\iota}p\right\}\right)\sum_{\nu=0}^{\infty} \frac{\left(\frac{\varsigma}{\iota}p\right)^\nu - \exp\{-(\iota + \varsigma)T\}\left(\exp\{-\iota T\}\frac{\varsigma}{\iota}p\right)^\nu}{(\nu + 1 + \varsigma/\iota)\nu!} \\
&\geq L\frac{1}{p}\exp\left\{\frac{\varsigma}{\iota}p\right\} - H\frac{1}{p}\exp\left\{\exp\{-\iota T\}\frac{\varsigma}{\iota}p\right\} \quad (2)
\end{aligned}$$

for some constants H, L we have

$$\mathbb{P}(F_1) \geq \frac{C_1}{p} - \frac{C_2}{\exp\{\xi p\} p} \tag{3}$$

for some positive constants C_1, C_2, ξ. Since, in addition, $F_1 \perp\!\!\!\perp F_2$ and since $\mathbb{P}(F_2)$ does not depend on p, we are finally getting that

$$0 < \lim_{p \to \infty} \frac{\mathbb{P}(a_i - a_0 > p)}{p}. \qquad i \in \mathbb{N}.$$

i.e. the tail index of a_{t_i} is indeed one.

Finally, since the sequence $\tau_1, \tau_2, \dots$ comprises all the times $t_1, t_2, \dots$, it may be easily shown that the properties of the exponents of a_{t_i} apply also to a_{τ_i} for any i.

5. Tails at a Fixed Horizon

To confirm our findings in "changed condition" let us examine, in addition to the "tick time", the price increment at a fixed horizo $T > 0$

Assume $\lambda > 0$ first; in this case,

$$\mathbb{P}(\Delta d_i > p | \Theta_i, t_i, t_{i+1}, \dots) \leq \exp\{-\gamma p\}, \qquad \gamma = \lambda \wedge \frac{\varsigma}{\iota}$$

(we have used the fact that $A_{t_i^-} \perp\!\!\!\perp t_{i+1}, t_{i+2}, \dots$) hence

$$\mathbb{P}(d_i - a_0 > p | t_1, t_2, \dots) \leq \int_p^\infty \epsilon_i(p) dp$$

where

$$\epsilon_i(p) = \frac{\gamma^i p^{i-1} e^{-\gamma p}}{(i-1)!}$$

is the p.d.f. of the Erlang distribution with parameters i and γ.

Because $I = \max\{i : t_i \leq T\}$ is Poisson with $\eta + \iota$ and since I is $\sigma(t_1, t_2, \dots)$-measurable, we have, for any $p > 1$,

$$\begin{aligned}
\mathbb{P}(d_I - a_0 > p) &= \sum_{i=1}^\infty \mathbb{P}(d_i - a_0 > p | I = i) \mathbb{P}(I = i) \\
&= \sum_{i=1}^\infty \mathbb{P}(d_i - a_0 > p | t_1, t_2, \dots) \mathbb{P}(I = i) \leq \int_p^\infty e^{-\eta - \iota - \gamma z} \sum_{i=1}^\infty \frac{(\eta + \iota)^i \gamma^i z^{i-1}}{i!(i-1)!} dz \\
&\leq \int_p^\infty e^{-\eta - \iota - \gamma z} \left(\sum_{i=1}^{2\lceil \eta + \iota \rceil} c_i z^i + C \sum_{i=0}^\infty \frac{(\gamma/2)^i z^i}{i!} \right) dz \\
&\leq \sum_{i=1}^{2\lceil \eta + \iota \rceil} \int_p^\infty e^{-\eta - \iota - \gamma z} c_i z^i dz + C \int_p^\infty e^{-\eta - \iota - (\gamma/2) z} dz
\end{aligned}$$

for some positive $C, c_1, \dots, c_{2\lceil \eta + \iota \rceil}$ - since all the summands on the r.h.s. vanish at the exponential rate, the tails of $a_T - a_0$ are thin.

Finally, let us prove that the tail index of a_T is at most one if $\lambda = 0$: since $a_T - a_0 > p$ if $\Delta d_1 > 2p$, $t_1 \leq T$, $t_2 \geq T$, $\vartheta \geq T$ and $\tau_2' \geq T$ we get, similarly to the previous subsection, that

$$\mathbb{P}(a_T > p) \geq \left(\frac{C_1}{p} - \frac{C_2 \exp\{-\xi p\}}{p} \right) \beta$$

for some $\beta > 0$ (see (2)) i.e.

$$\frac{\lim_{p \to \infty} \mathbb{P}(a_T > p)}{p} \geq 0$$

proving that the tail exponent of a_T is at most one.

6. Conclusion

We have described the distribution of the model by Smith et al. [2003] and we studied the tail behavior of its price increments. We found the tail exponent to be one if the initial order book is empty but infinity if an initial call auction is held at the start of the trading.

Concluding the paper, let us stress that we are not in any contradiction with papers finding greater tail exponents at similar models (e.g. Slanina [2001]) because, contrary to them, we do not study tails of the stationary distributions; since, by our computations, the weights of tails of order one decrease with the increasing time, our results in fact support the hypothesis that the tails of stationary price increments are lighter than one. In this light, our "fat-tailed" result does not seem to be any revolutionary one. What appears quite surprising, on the other hand, is the demonstrated stabilizing role of the initial call auction.

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