Complex Economic Systems in Macro & Finance
Lecture I: Complex Systems, Nonlinear Dynamics, Behavioral Rationality and Heterogeneous Expectations

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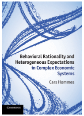
Outline of the Workshop: 4 Lectures

1. Complex systems, nonlinear dynamics, behavioral rationality, heterogeneous expectations
2. Asset pricing model with heterogeneous beliefs
3. Empirical validation and Laboratory Experiments
4. Behavioral macroeconomics with heterogeneous expectations

Some References

- Hommes, C.H., (2013), Behavioral Rationality and Heterogeneous Expectations in Complex Economic Systems, Cambridge.



  - complex nonlinear dynamics: Chapters 2 (1-D) and 3 (2-D + higher D)
  - nonlinear cobweb model with homogeneous expectations (Chapter 4)
  - cobweb model with heterogeneous expectations (rational versus naive) (Chapter 5)
  - asset pricing model with heterogeneous expectations (Chapter 6)
  - empirical validation (Chapter 7)
  - laboratory experiments (Chapter 8)
  - some more recent papers on behavioral macro and housing market

What is a Complex System

Characteristic Features

- **nonlinear**, complicated dynamics (multiple steady states, cycles, chaos)
- **interacting** particles/agents
- **emergent macro behavior** may be different from **individual micro behavior**
- **heterogeneous** agents, different from representative agent
- **complex socio-economic systems**: the particles can think
  need a theory of adaptive behavior
- simple behavioral **heuristics** instead of perfect rationality

Outline

Quadratic Map

Example of one-dimensional system:

$$x_{t+1} = f_{\lambda(x_t)} = \lambda x_t(1 - x_t) \tag{2.2}$$

1. initial state $x_0 \in [0, 1]$
2. **parameter** λ, $0 \leq \lambda \leq 4$.
3. **Problem**: what do the **orbits** look like?

Convergence to a steady state

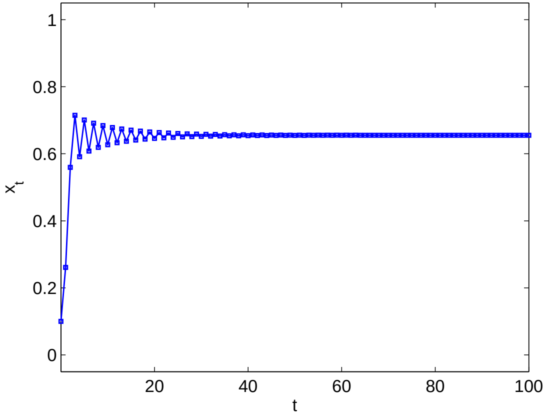


Figure : $\lambda = 2.9$ and $x_0 = 0.1$

Convergence to a 2-cycle

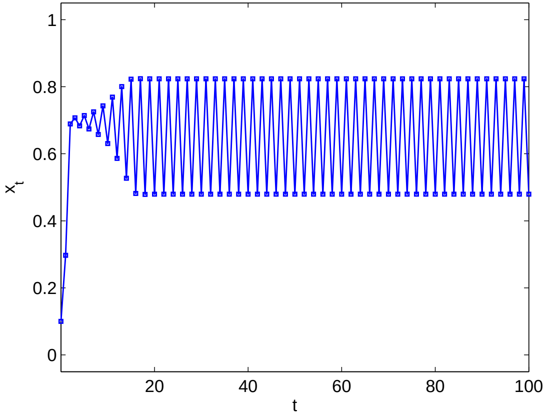


Figure : $\lambda = 3.3$ and $x_0 = 0.1$

Convergence to a 4-cycle

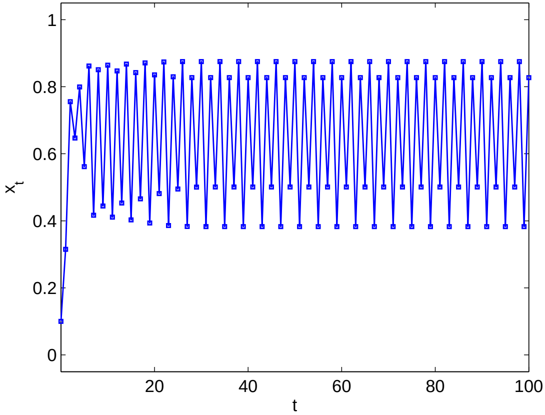


Figure : $\lambda = 3.5$ and $x_0 = 0.1$

Convergence to a 3-cycle

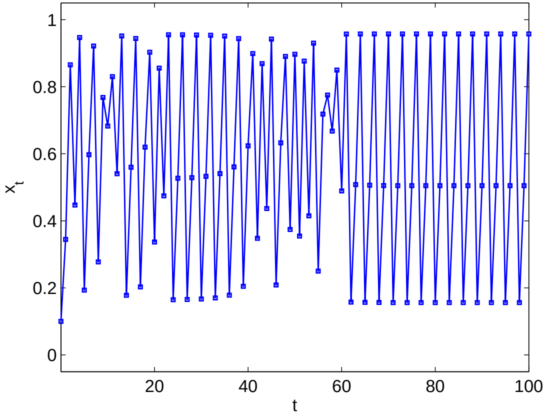


Figure : $\lambda = 3.83$ and $x_0 = 0.1$

Sensitive dependence on initial conditions

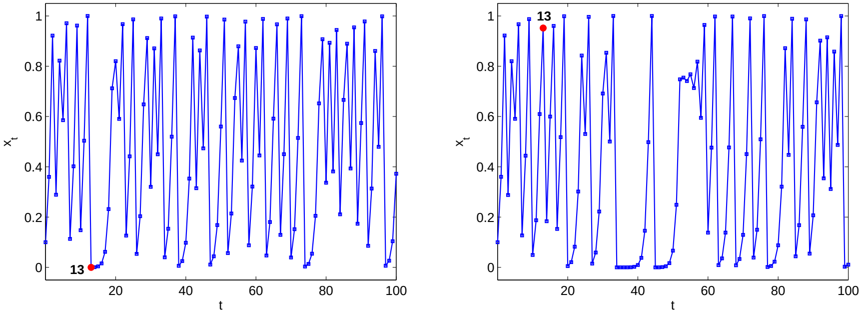


Figure : $\lambda = 4$ and $x_0 = 0.1$ (left) and (f) $\lambda = 4$ and $x_0 = 0.1001$ (right).

Periodic Orbits and Stability

A point x is called a **periodic point with period k** if

$$f^k(x) = x \quad \text{and} \quad f^i(x) \neq x, \quad 0 < i < k.$$

(Note: periodic point with period k is fixed point of k-th iterate f^k)

$\{x_1, x_2, ..., x_k\} = \{x_1, f(x_1), f^2(x_1), ...f^{k-1}(x_1)\}$ **periodic orbit** or **k-cycle**.

If x_i stable fixed point of f^k, then $\{x_1, x_2, ..., x_k\}$ **stable periodic orbit**; from the chain rule we have

$$\begin{aligned}(f^k)'(x_i) = (f^k)'(x_1) &= f'(f^{k-1}(x_1)) \cdot f'(f^{k-2}(x_1)) .... f'(f(x_1)) \cdot f'(x_1) \\ &= \prod_{i=0}^{k-1} f'(f^i(x_1)).\end{aligned}$$

(Note: $(f^k)'(x_j)$ is the product of derivatives along the orbit)

Aperiodic Point

A point x is called an **aperiodic point** if

1. the orbit of x is bounded,
2. the orbit of x is not periodic, and
3. the orbit of x does not converge to a periodic orbit

Bifurcation diagram of the quadratic map

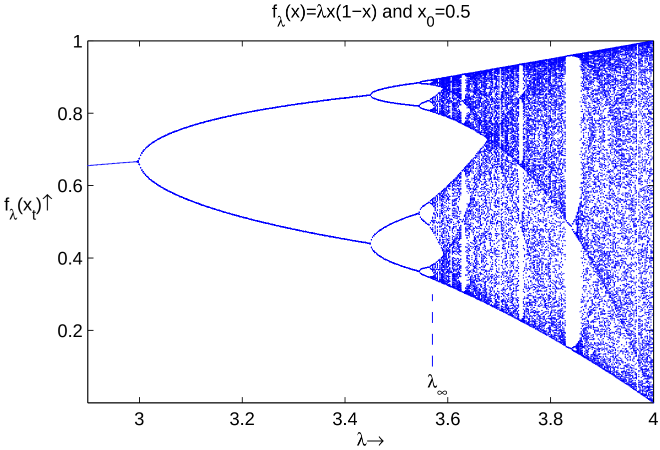


Figure : Bifurcation diagram of the quadratic map.

Period doubling bifurcation at $\lambda = 3$

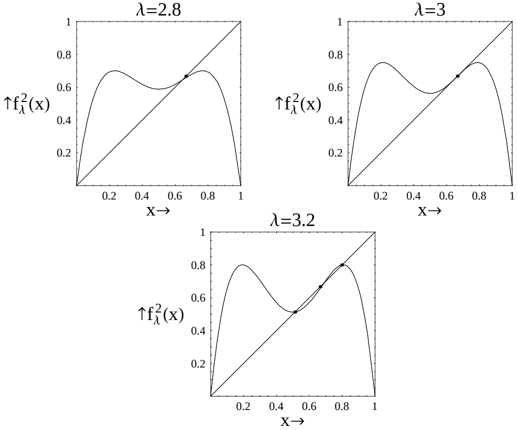


Figure : Graphs of the second iterate f^2 for three different λ-values close to the period-doubling bifurcation at $\lambda = 3$.

Creation of a 3-cycle by tangent bifurcation

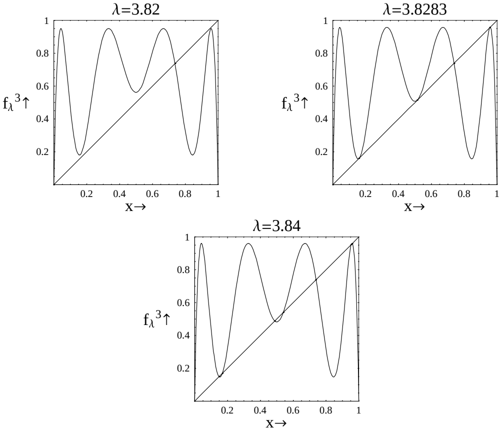


Figure : Creation of 3-cycle by tangent bifurcation at $\lambda \approx 3.8283$.

Definition of topoligical chaos

The dynamics of a difference equation $x_{t+1} = f(x_t)$ is called **(topologically) chaotic** if the following three properties are satisfied:

1. There exists an infinite set P of (unstable) periodic points with different periods.
2. There exists an uncountable set S of aperiodic points (i.e. poinst whose orbits are bounded, not periodic and not converging to a periodic orbit).
3. f has sensitive dependence on initial conditions w.r.t. $\Lambda = P \cup S$, that is, there exists a positive distance C such that for all initial states $x_0 \in \Lambda$ and any ε-neighbourhood U of x_0, there exists an initial state $y_0 \in \Lambda \cap U$ and a time $T > 0$ such that the distance $d(x_T, y_T) = d(f^T(x_0), f^T(y_0)) > C$.

Example of Chaos: quadratic map for $\lambda = 4$

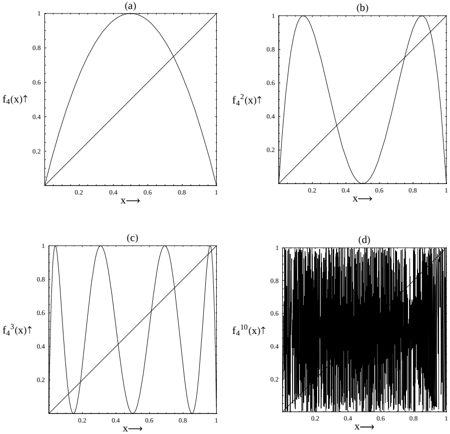


Figure : Graphs of (a) $f_4(x) = 4x(1-x)$, (b) f_4^2, (c) f_4^3 and (d) f_4^{10}.

Properties of quadratic map f_4

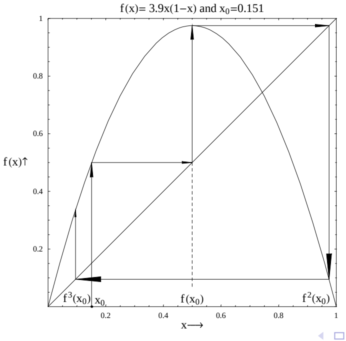
It can be shown that for any n, the graph of f_4^n has the following properties:

1. f_4^n has 2^{n-1} maxima equal to 1 and $2^{n-1}+1$ minima equal to 0 (including minima at $x=0$ and $x=1$).
2. f_4^n 'oscillates' 2^{n-1} times on the interval $[0,1]$.
3. the map f_4^n has 2^n fixed points.
4. for any interval I of arbitrarily small length ε, there exists an $N>0$ such that I contains points x, y with $f_4^N(x)=0$ and $f_4^N(y)=1$.

Period three implies chaos

Theorem 1

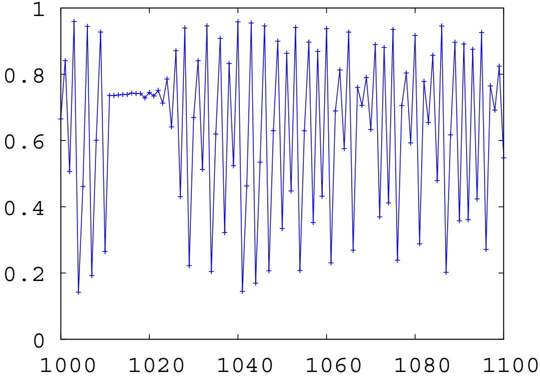
("Period 3 implies Chaos", Li & Yorke [1975]). Let $x_{t+1} = f(x_t)$ be a 1-D difference equation with f a continuous map. If there exist a point x_0 such that $f^3(x_0) \leq x_0 < f(x_0) < f^2(x_0)$ (or with $>$ instead of $<$) then the dynamics is topologically chaotic



Topological Chaos with Noise

Quadratic map with small noise

$$x_{t+1} = 3.83x_t(1 - x_t)$$



Lyapunov exponents

- The **Lyapunov exponent** is defined as $\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln(\mid f'(f^i(x_0)) \mid)$.
- Derivation:

$$\mid f^n(x_0+\delta) - f^n(x_0) \mid \approx \mid (f^n)'(x_0)\delta \mid = \; e^{n\lambda(x_0)} \mid \delta \mid$$

$$\Leftrightarrow e^{n\lambda(x_0)} = \mid (f^n)'(x_0) \mid \Rightarrow \lambda(x_0) = \frac{1}{n}\ln(\mid (f^n)'(x_0) \mid).$$

- The Lyapunov exponent measures the **average rate of divergence of nearby initial states**. It is the average of the logs of the absolute values of the derivative along the orbit.

Lyapunov exponent plot of the quadratic map

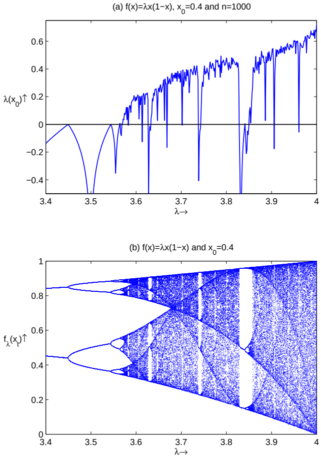


Figure : Lyapunov exponent L as a function of the parameter λ.

Two-dimensional (2-D) systems

$$(x_{t+1}, y_{t+1}) = F_\lambda(x_t, y_t),$$

F_λ nonlinear 2-D map and λ is a parameter.
The **orbit** with **initial state** (x_0, y_0) is the set

$$\{(x_0, y_0), (x_1, y_1), (x_2, y_2), ...\} = \{(x_0, y_0), F_\lambda(x_0, y_0), F_\lambda^2(x_0, y_0), ...\}.$$

Problem: what do these orbits look like and how does it depend on initial states and parameters?
Example: Hénon map:

$$\begin{aligned} x_{t+1} &= 1 - ax_t^2 + y_t \\ y_{t+1} &= bx_t, \end{aligned}$$

where a and b are parameters.
(special case $b = 0$ yields 1-D quadratic map)

Strange attractor of the Hénon map

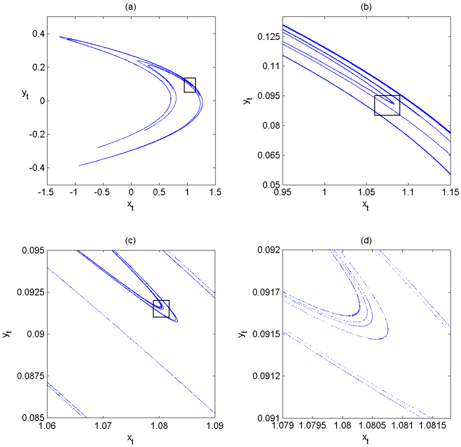


Figure : The strange attractor for the Hénon map $H_{a,b}$ with $a = 1.4$ and $b = 0.3$.

Chaotic time-series and SDIC

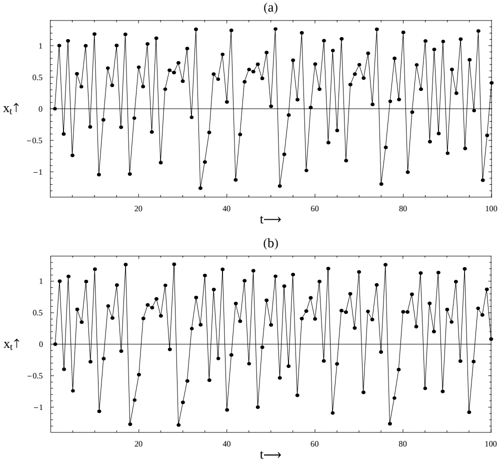


Figure : Chaotic time series and sensitive dependence for the Hénon map $H_{a,b}$ with $a = 1.4$ and $b = 0.3$. (a) $(x_0, y_0) = (0, 0)$ and (b) $(x_0, y_0) = (0.001, 0)$.

Attractor and Strange Attractor

An **attractor** of a K-dimensional system $X_{t+1} = F(X_t)$ is a compact set A with the following properties:

1. The set A is **invariant**, i.e. $F(A) \subset A$.
2. The set A is **invariant**, i.e. there exists an open neighborhood U of A (i.e. $A \subset U$), such that all initial states X_0 converge to the attractor A, i.e. for all $X_0 \in U$, $\lim_{n\to\infty} dist(F^n(X_0), A) = 0$.
3. There exists an initial state $X_0 \in A$ for which the orbit is **dense** in A.

An attractor A is called a **strange attractor** of the N-dimensional dynamical system $x_{t+1} = F(x_t)$, if the map F has sensitive dependence w.r.t. the set of initial states converging to A.

The delayed logistic map

- Delayed logistic map: $N_{t+1} = aN_t(1 - N_{t-1})$.
- Equivalently ($x_t = N_t$ and $y_t = N_{t-1}$):

$$\begin{aligned} x_{t+1} &= y_t \\ y_{t+1} &= ay_t(1 - x_t). \end{aligned}$$

- **steady states**

$$(x_1, y_1) = (0, 0) \qquad \text{and} \qquad (x_2, y_2) = (\frac{a-1}{a}, \frac{a-1}{a}).$$

- The **eigenvalues** of the system are $\lambda_1 = \frac{1}{2} - \frac{1}{2}\sqrt{5 - 4a}$ and $\lambda_2 = \frac{1}{2} + \frac{1}{2}\sqrt{5 - 4a}$.

Dynamical properties of the delayed logistic map

The eigenvalues λ_1 and λ_2 of $JF_a(\frac{a-1}{a}, \frac{a-1}{a})$ satisfy the following properties:

- $0 \leq a < 1$: real eigenvalues with $-1 < \lambda_1 < 1 < \lambda_2$, so $(\frac{a-1}{a}, \frac{a-1}{a})$ is a **saddle**.
- $1 < a < \frac{5}{4}$: real eigenvalues with $0 < \lambda_1 < \lambda_2 < 1$, so $(\frac{a-1}{a}, \frac{a-1}{a})$ is attracting (**stable node**).
- $\frac{5}{4} < a < 2$: complex eigenvalues with $\lambda_1 \lambda_2 = a - 1 < 1$, so $(\frac{a-1}{a}, \frac{a-1}{a})$ is a **stable focus**.
- $a > 2$: complex eigenvalues with $\lambda_1 \lambda_2 = a - 1 > 1$, so $(\frac{a-1}{a}, \frac{a-1}{a})$ is an **unstable focus**.
- **Hopf bifurcation** (or Neimark-Sacker) for $a = 2$
  complex eigenvalues on the unit circle

Attractors delayed logistic map

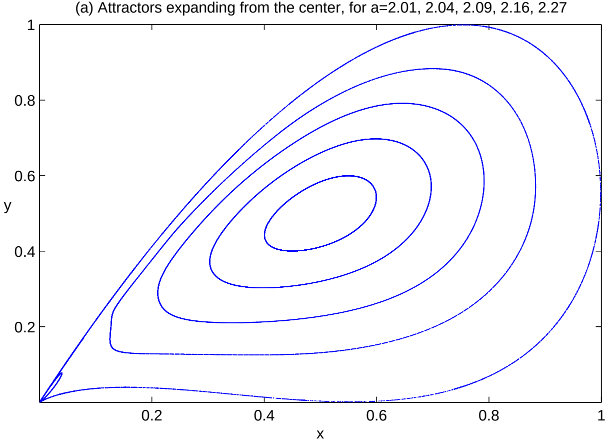


Figure : (a) Attractors for the logistic delayed equation for different a-values after the Hopf bifurcation.

Strange attractor of the delayed logisitic map

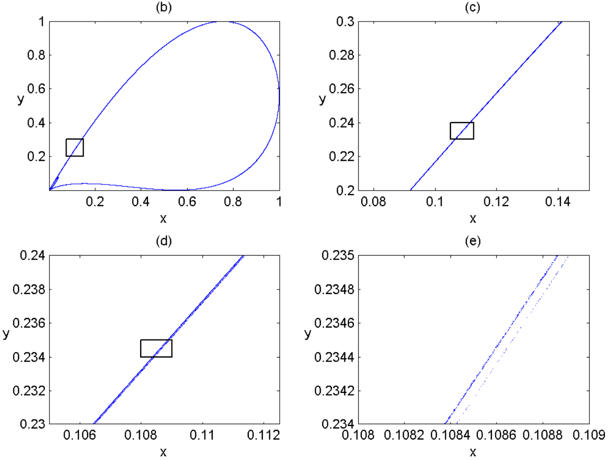


Figure : (b-e) The strange attractor for $a = 2.27$ and some enlargements.

Time series for the logistic delayed equation

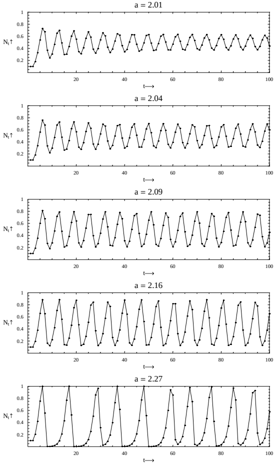


Figure : Time series for for different values of the parameter a.

Bifurcation diagrams for logistic delayed equation

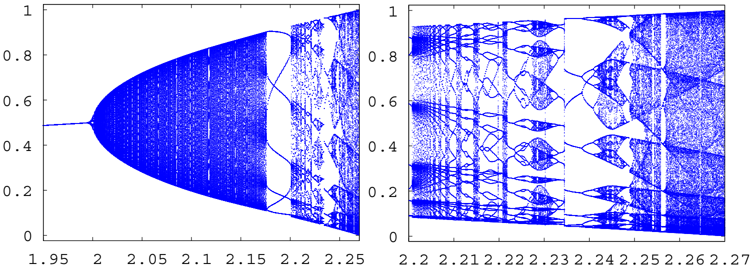


Figure : **Hopf-bifurcation** and **breaking of an invariant circle** bifurcation route to chaos

Types of Local Bifurcations

1. $\lambda = +1$:
   - **saddle-node bifurcation**: two new steady states, a saddle and a node;
   - **pitchfork bifurcation**: one steady state becomes unstable and two new stable steady states;
   - **transcritical bifurcation**; two steady states collide and exchange stability;
2. $\lambda = -1$:
   - **period-doubling bifurcation**: steady state loses stability and new stable 2-cycle;
3. a pair of complex eigenvalues λ_1 and λ_2 on the unit circle, i.e. $|\lambda_1 \lambda_2| = 1$:
   - **Hopf bifurcation**: steady state becomes an unstable focus and an attracting invariant circle emerges with (quasi-)periodic dynamics

Local (un)stable manifolds

Let p be a fixed point of the 2-D map F.
The **local stable manifold** and **local unstable manifold** of p are defined as

$$W^s_{loc}(p) = \{x \in U | \lim_{n\to\infty} F^n(x) = p\} \tag{1}$$

$$W^u_{loc}(p) = \{x \in U | \lim_{n\to-\infty} F^n(x) = p\} \tag{2}$$

where U is some small neighbourhood of p.

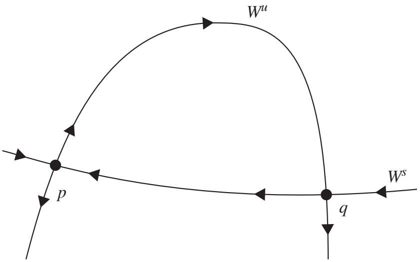
Global(un)stable manifolds

The **global stable manifold** and the **global unstable manifold** are now defined as

$$W^s(p) = \bigcup_{n=0}^{\infty} F^{-n}(W^s_{loc}) \tag{3}$$

$$W^u(p) = \bigcup_{n=0}^{\infty} F^{n}(W^u_{loc}). \tag{4}$$

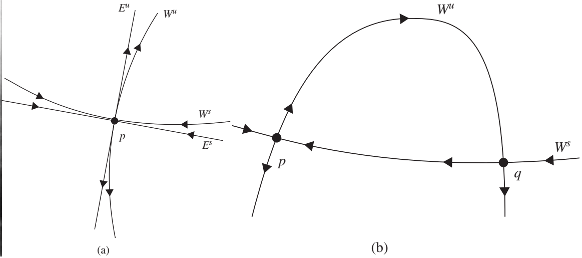
Homoclonic point



(b)

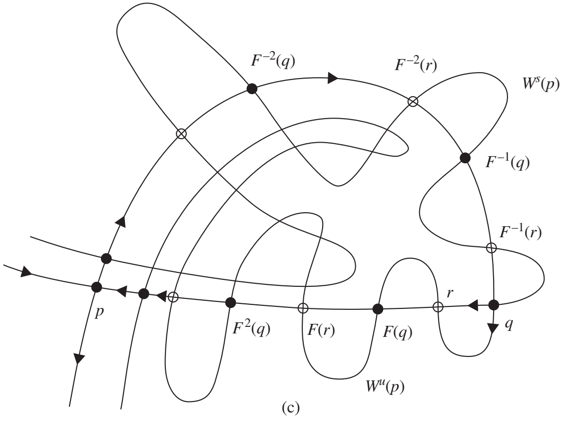
A point q is called a **homoclinic point** if $q \neq p$ and q is an intersection point of the stable and unstable manifolds of the saddle point p, that is, $q \in W^s(p) \bigcap W^u(p)$.

Henry Poincaré, ca. 1890: motion in the **three body problem** is unpredictable and chaotic



2-D map of suitable plane section has **homoclinic orbit**

Henry Poincaré, ca. 1890: homoclinic orbit implies sensitive dependence on initial conditions



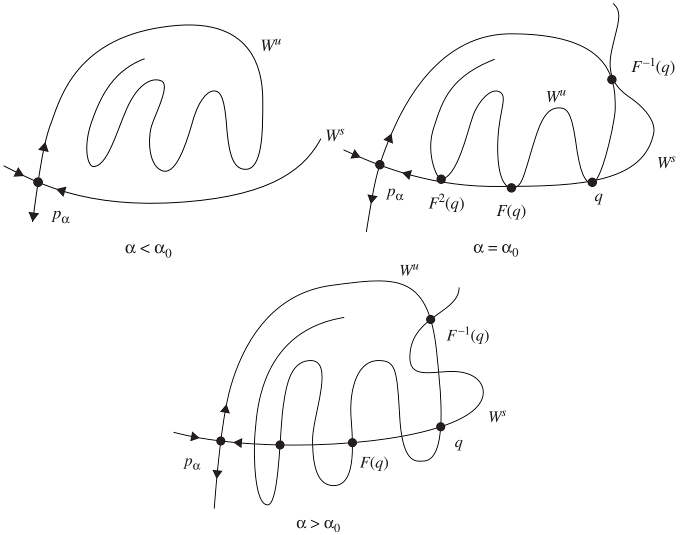
(c)

Homoclonic bifurcation

We say that F_α has a **homoclinic bifurcation**, associated to the saddle point p_α, at $\alpha = \alpha_0$, if

1. for $\alpha < \alpha_0$, $W^s(p_\alpha)$ and $W^u(p_\alpha)$ have no intersection point $q \neq p$;
2. for $\alpha = \alpha_0$, $W^s(p_\alpha)$ and $W^u(p_\alpha)$ have a point of homoclinic tangency;
3. for $\alpha > \alpha_0$, $W^s(p_\alpha)$ and $W^u(p_\alpha)$ have a transversal homoclinic intersection point.

Homoclonic bifurcation



Implications Nonlinear Dynamics for Economics

- The fact that (simple) nonlinear systems exhibit **complex dynamics** calls for **reservations about rational behavior**, in particular **rational expectations**;
- In a **nonlinear** world, simple **heuristics** that work reasonably well may be the best **boundedly rational** agents can achieve