Complex Economic Systems in Macro & Finance
Lecture 2: Behavioural Asset Pricing Model with Heterogeneous Expectations

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Outline

1. Introduction
2. The model
   - 2-type model: fundamentalists vs. trend followers
   - 3-type model: fundamentalists vs. optimists and pessimists
   - 4-type model: fundamentalists vs. trend and bias
3. Empirical Validation

Literature

- Hommes, C.H., (2013), Behavioral Rationality and Heterogeneous Expectations in Complex Economic Systems, Cambridge.
  (**Chapters 6 and 7**)
- W.A. Brock and C.H. Hommes, A rational route to randomness, *Econometrica* 65, (1997), 1059-1095.
- W.A. Brock and C.H. Hommes, Heterogeneous beliefs and routes to chaos in a simple asset pricing model, *Journal of Economic Dynamics and Control* 22, (1998), 1235-1274.
- Hommes, C.H. (2006), Heterogeneous Agent Models In Economics and Finance, In: Handbook of Computational Economics, Volume 2: Agent-Based Computational Economics, Edited by L. Tesfatsion and K.L. Judd, Elsevier Science B.V., 2006, 1109-1186.
- Hommes, C.H. and in't Veld, D. (2014), Behavioral heterogeneity and the financial crisis, CeNDEF working paper, July 2014.

Traditional Rational View

- **representative** agent, who is perfectly **rational**
- expectations are **model consistent**
- **Friedman hypothesis**: "irrational agents will lose money and **will be driven out** the market by rational agents"
- simple (linear), stable model, driven by **exogenous random news** about fundamentals
- prices reflect economic fundamentals (**market efficiency**)
- **Lucas**: **macroeconomic policy** should be based on rational expectations

Heterogeneous, interacting agents approach

- **heterogeneous** agents, heterogeneous beliefs
- market **psychology**, **herding** behavior (Keynes (1936))
- **bounded rationality** (Simon (1957))
- markets as **complex adaptive, nonlinear evolutionary** systems
- **interactions** of agents create **aggregate structure** explaining stylized facts

Some Problems Interacting Agents Approach

- ‘wilderness’ of bounded rationality
- many degrees of freedom for heterogeneity
- what exactly causes the outcome in a (large) computational HAM

How to Discipline Bounded Rationality?

- **stylized** agent-based models
- **behavioral rationality –behavioral consistency**: simple heuristics that work reasonably well
- **evolutionary selection** (‘survival of the fittest’) and reinforcement **learning**
- **laboratory experiments** to test individual decision rules and aggregate macro behavior

Asset Pricing Model with Homogeneous Beliefs

Agents choose between **risk free** and **risky** asset:

- $R = 1 + r > 1$: gross return on risk free asset
- p_t: price (ex div.) per share of risky asset
- y_t: IID dividend process for risky asset
- z_t: number of shares purchased at date t

End of period wealth:

$$\mathbf{W}_{t+1} = R(W_t - p_t z_t) + (\mathbf{p}_{t+1} + \mathbf{y}_{t+1})z_t = RW_t + (\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - Rp_t)z_t$$

Myopic mean variance maximization: demand z_t solves

$$Max\{E_t\mathbf{W}_{t+1} - \frac{a}{2}V_t\mathbf{W}_{t+1}\}, \quad \text{so}$$

$$z_t = \frac{E_t[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - Rp_t]}{aV_t[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - Rp_t]} = \frac{E_t[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - Rp_t]}{a\sigma^2}$$

(where σ^2 is common beliefs on variance V_{ht}).

Equilibrium Price

Market equilibrium between supply and demand:

$$\frac{E_t(\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - Rp_t)}{a\sigma^2} = z^s$$

equilibrium pricing equation:

$$Rp_t = E_{ht}(\mathbf{p}_{t+1} + \mathbf{y}_{t+1}) - a\sigma^2 z^s$$

special case: constant zero supply of outside shares $z^s = 0$:

$$Rp_t = E_{ht}(\mathbf{p}_{t+1} + \mathbf{y}_{t+1})$$

REE fundamental solution

Equilibrium pricing equation:
(common beliefs on future dividends $E_t[y_{t+1}]$)

$$Rp_t = E_t[\mathbf{p}_{t+1} + \mathbf{y}_{t+1}]$$

“no bubble" condition implies unique bounded **fundamental solution** p_t^*:
(discounted sum expected future cash flow)

$$p_t^* = \frac{E_t[y_{t+1}]}{R} + \frac{E_t[y_{t+2}]}{R^2} + \cdots$$

For special case of IID dividends, with $E_t[y_{t+1}] = \bar{y}$:

$$p^* = \frac{\bar{y}}{R-1} = \frac{\bar{y}}{r}$$

Model in deviations from fundamental

deviation from fundamental

$$x_t = p_t - p^*$$

Pricing equation in deviations:

$$Rx_t = E_t x_{t+1}$$

Notice: **rational bubble** solutions: $x_t = x_0 R^t$
with **self-fulfilling** belief $x^e_{t+1} = g x_{t-1}$, with $g = R^2$.

Equilibrium pricing equation with **heterogeneous beliefs**:
(in deviations from RE-fundamental)

$$Rx_t = \sum_{h=1}^{H} n_{ht} E_{ht} x_{t+1}$$

Asset pricing model with heterogeneous beliefs

agents choose to invest in **risk free** or **risky** asset

- $R = 1 + r > 1$: gross return on risk free asset
- p_t: price (ex div.) per share of risky asset
- y_t: IID dividend process for risky asset
- n_{ht} : fraction of agents of type h

Myopic mean variance maximization of expected wealth
demand for risky asset by type h:

$$z_{ht} = \frac{E_{ht}[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - Rp_t]}{aV_{ht}[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - Rp_t]} = \frac{E_{ht}[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - Rp_t]}{a\sigma^2}$$

(common beliefs on variance $V_{ht} = \sigma^2$ and a risk aversion parameter)

Market equilibrium

Equilibrium of supply and demand:

$$\sum_{h=1}^{H} n_{ht} \frac{E_{ht}[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - Rp_t]}{a\sigma^2} = z^s$$

equilibrium pricing equation:

$$Rp_t = \sum_{h=1}^{H} n_{ht} E_{ht}(\mathbf{p}_{t+1} + \mathbf{y}_{t+1}) - a\sigma^2 z^s$$

special case: constant zero supply of outside shares $z^s = 0$:

$$Rp_t = \sum_{h=1}^{H} n_{ht} E_{ht}(\mathbf{p}_{t+1} + \mathbf{y}_{t+1}) + \epsilon_t$$

(where noise term ϵ_t (e.g. random supply of shares) has been added)

Heterogeneous Beliefs

Assumptions about beliefs of trader type h:

B1 same constant beliefs on variances for all types h:
$V_{ht}[\mathbf{p}_{t+1}+\mathbf{y}_{t+1} - Rp_t] = V_t[\mathbf{p}_{t+1}+\mathbf{y}_{t+1} - Rp_t] = \sigma^2$

B2 common and correct beliefs on future dividends:
$E_{ht}[\mathbf{y}_{t+1}] = E_t[\mathbf{y}_{t+1}]$, for all types h
special case of IID dividends:
$E_{ht}[\mathbf{y}_{t+1}] = E_t[\mathbf{y}_{t+1}] = \bar{y}$.

B3 **heterogeneous beliefs** on future prices of the form:
$E_{ht}[\mathbf{p}_{t+1}] = E_t[\mathbf{p}^*_{t+1}] + E_{ht}[\mathbf{x}_{t+1}] = p^*_{t+1} + f_h(x_{t-1}, \cdots x_{t-L})$
special case of IID dividends:
$E_{ht}[p_{t+1}] = p^* + f_h(x_{t-1}, \cdots x_{t-L})$

Under assumptions B1-B3 **equilibrium pricing equation** in **deviations** $x_t = p_t - p^*$ from the fundamental:

$$Rx_t = \sum_{h=1}^{H} n_{ht} E_{ht}[\mathbf{x}_{t+1}] = \sum_{h=1}^{H} n_{ht} f_{ht}$$

Forecasting rules

belief of type h on future prices:

$$E_{ht}[p_{t+1}] = p^* + f_h(x_{t-1}, ..., x_{t-L})$$

or in deviations:

$$E_{ht}[x_{t+1}] = f_h(x_{t-1}, ..., x_{t-L})$$

Important special cases:

- **rational expectations**: "$f(x_{t-1}, ..., x_{t-L})$" $= x_{t+1}$ (also **perfect foresight** on other belief-fractions n_{ht})
- **fundamentalists**: $f \equiv 0$ (no knowledge about other beliefs and fractions n_{ht})
- pure **trend chasers**: $f(x_{t-1}, ..., x_{t-L}) = gx_{t-1}$
- pure **bias**: $f(x_{t-1}, ..., x_{t-L}) = b$.
- **simple example**: linear forecast with one lag $f_{ht} = g_h x_{t-1} + b_h$
- **trend extrapolator**: $f_{ht} = x_{t-1} + g_h(x_{t-1} - x_{t-2})$

Question: Do rational agents and/or fundamentalists drive out trend chasers and biased beliefs?

Evolutionary selection of strategies

evolutionary selection or **reinforcement learning**:
more successful strategies attract more followers

fractions of belief types are updated in each period, according to (discrete choice model, BH 1997, 1998)

$$n_{ht} = \frac{e^{\beta U_{h,t-1}}}{Z_{t-1}}$$

where Z_{t-1} is normalization factor and β is **intensity of choice**.
$\beta = 0$: all types equal weight
$\beta = \infty$: "neoclassical limit", i.e. all agents use best predictor

Evolutionary selection of strategies

realized profits in period t

$$\pi_{ht} = R_t z_{h,t-1} = (p_t + y_t - Rp_{t-1})\frac{E_{h,t-1}[p_t + y_t - Rp_{t-1}]}{a\sigma^2}$$

$$(x_t - Rx_{t-1} + \delta_t)\frac{E_{h,t-1}[x_t - Rx_{t-1}]}{a\sigma^2}$$

(with $y_t = \bar{y} + \delta_t$)

Fitness function or **performance measure**

(weighted sum of) realized profits

$$U_{ht} = \pi_{ht} + wU_{h,t-1} - C_h$$

where $C_h \geq 0$ are **costs** for predictor h, and

w is **memory strength**

($w = 1$: infinite memory; fitness $\equiv$ accumulated wealth

$w = 0$: memory one lag; fitness $\equiv$ most recently realized net profit)

Asset Pricing Model with Heterogeneous Beliefs

(in deviations from the RE fundamental)

$$R x_t = \sum_{h=1}^{H} n_{ht} f_h(x_{t-1}, ..., x_{t-L}) = \sum_{h=1}^{H} n_{ht} f_{ht}$$

$$n_{ht} = \frac{e^{\beta U_{h,t-1}}}{\sum_{h=1}^{H} e^{\beta U_{h,t-1}}}$$

$$U_{h,t-1} = (x_{t-1} - R x_{t-2}) \frac{f_{h,t-2} - R x_{t-2}}{a \sigma^2} - C_h$$

Examples with Heterogeneous Linear Beliefs

(in deviations from the RE fundamental)
example with linear predictors $f_{ht} = g_h x_{t-1} + b_h$:

$$Rx_t = \sum_{h=1}^{H} n_{ht}(g_h x_{t-1} + b_h)$$

$$n_{ht} = \frac{e^{\beta U_{h,t-1}}}{\sum_{h=1}^{H} e^{\beta U_{h,t-1}}}$$

$$U_{h,t-1} = (x_{t-1} - Rx_{t-2})\frac{(g_h x_{t-3} + b_h - Rx_{t-2})}{a\sigma^2} - C_h$$

Two-type Example: Fundamentalists versus trend

Two trader types, with forecasting rules

$$
\begin{aligned}
f_{1t} &= 0, && \text{fundamentalists at costs C} \\
f_{2t} &= g x_{t-1}, \quad g > 0, && \text{trend followers}
\end{aligned}
$$

Define difference in fractions: $m_t = n_{1t} - n_{2t}$

$$
\begin{aligned}
R x_t &= \tfrac{1-m_t}{2} g x_{t-1} \\
m_{t+1} &= \tanh\Big(\frac{\beta}{2}\Big[-\frac{g x_{t-2}}{a\sigma^2}(x_t - R x_{t-1}) - C\Big]\Big)
\end{aligned}
$$

Two-type Example: Fundamentalists versus trend

Theorem. (existence and stability steady states)

$$m^{eq} = \tanh(-\beta C/2)$$
$$m^* = 1 - 2R/g$$

Let x^* be positive solution of

$$\tanh(\frac{\beta}{2}[\frac{g}{a\sigma^2}(R-1)(x^*)^2 - C]) = m^*$$

1. $0 < g < R$: $E_1 = (0, m^{eq})$ globally **stable** steady state
2. very strong trend chaser, i.e $g > 2R$: **three** steady states $E_1 = (0, m^{eq})$, $E_2 = (x^*, m^*)$ and $E_3 = (-x^*, m^*)$
3. $R < g < 2R$: if costs $C > 0$, then we have a **pitchfork bifurcation** for $\beta = \beta^*$, that is, a unique steady state for $\beta < \beta^*$ and three steady states for $\beta > \beta^*$.

Two-type Example: Fundamentalists versus trend

Moreover, **Hopf bifurcation** of non-fundamental steady states E_2 and E_3:

- stable for $\beta^* < \beta < \beta^{**}$
- unstable for $\beta > \beta^{**}$

Corollary: costly fundamentalists and cannot drive out trend chasers driven by short run profits.

Bifurcation diagram and Lyapunov exponent plot

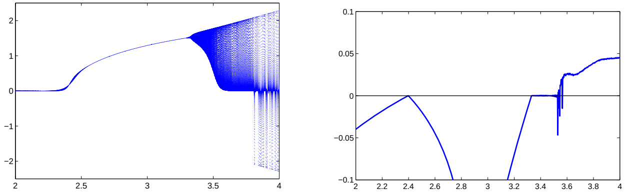


Figure : Bifurcation diagram (left) and largest Lyapunov exponent plot (right) for 2-type model with costly fundamentalist versus trend followers.

Time series of prices and fractions and attractors

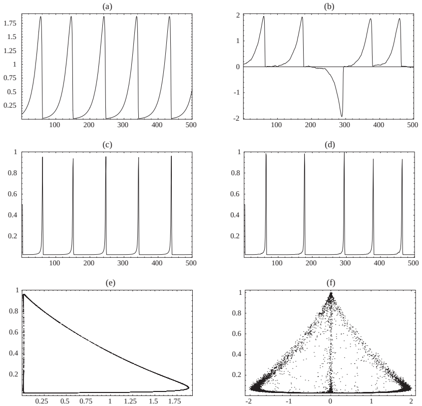


Figure : Time series of prices and fractions and attractors in the phase space for 2-type model with costly fundamentalist versus trend followers.

Homoclinic Orbit for $\beta = +\infty$

Two possibilities for the unstable manifold $W^u(E)$ (Brock and Hommes, 1998, Lemma 4, p.1251):

1. if $g > (1+r)^2$, then unstable manifold $W^u(E)$ equals the (**unbounded**) **unstable eigenvector**; typical solutions are exploding, diverging to infinity with trend followers dominating the market;
2. if $1 + r < g < (1+r)^2$, then unstable manifold $W^u(E)$ is **bounded**; all time paths converge to the (locally) unstable saddle-point fundamental steady state;
   homoclinic orbits

Three Type Example

Three types (zero costs)

$$
\begin{array}{lllll}
f_{1t} & = & 0 & & \text{fundamentalists} \\
f_{2t} & = & b & b > 0, & \text{positive bias (optimists)} \\
f_{3t} & = & -b & -b < 0, & \text{negative bias (pessimists).}
\end{array}
$$

$$
\begin{aligned}
Rx_t &= n_{2,t} b_2 + n_{3,t} b_3 \\
n_{j,t+1} &= \exp\left(\frac{\beta}{a\sigma^2}(b_j - Rx_{t-1})(x_t - Rx_{t-1})\right)/Z_t, \quad j = 1, 2, 3
\end{aligned}
$$

Three Type Example

Theorem. (existence and stability steady state)
Assume opposite bias, i.e. $b_2 > 0 > b_3$, then (26) has unique steady state E, which equals the fundamental steady state when $b_2 = -b_3$.
E exhibits a *Hopf bifurcation* for $\beta = \beta^*$:
E stable for $0 < \beta < \beta^*$ and
E unstable for $\beta > \beta^*$.
Corollary: Biased beliefs lead to a different route to complexity

Three Type Example

Theorem (neoclassical limit, i.e. $\beta = \infty$)
When biased beliefs are exactly opposite, i.e. $b_2 = -b_3 = b > 0$, then (26) has *globally stable* 4-*cycle*.
For all three types, average profit along this 4-cycle is b^2.
Corrolary
Fundemantalists with zero costs and infinite memory can not beat opposite biased beliefs!

Bifurcation diagram and largest Lyapunov exponent plot

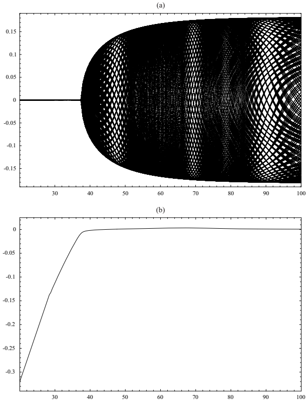


Figure : Bifurcation diagram (top panel) and largest Lyapunov exponent plot (bottom panel) for 3-type model with fundamentalists versus optimists and pessimists.

Phase plot

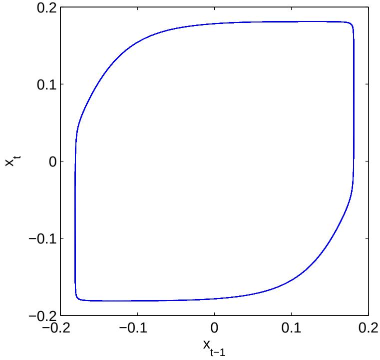


Figure : Phase plot for 3-type model with fundamentalists versus optimists and pessimists, as in Brock and Hommes, 1998, Figures 7 and 8.

Time series

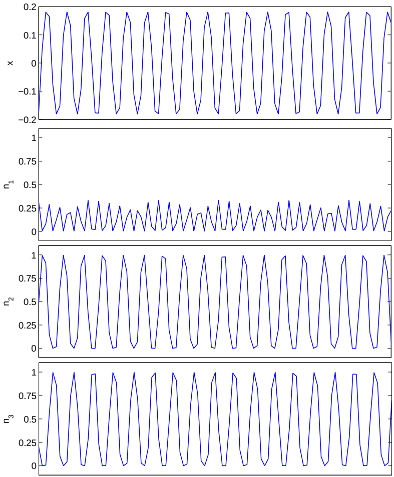


Figure : Time series for 3-type model with fundamentalists versus optimists and pessimists, as in Brock and Hommes, 1998, Figures 7 and 8.

Four Belief Types

(zero costs; memory one lag)
example.

$$\begin{array}{lllllll} g_1 & = & 0 & b_1 & = & 0 & \text{fundamentalists} \\ g_2 & = & 0.9 & b_2 & = & 0.2 & \text{trend + upward bias} \\ g_3 & = & 0.9 & b_3 & = & -0.2 & \text{trend + downward bias} \\ g_4 & = & R = 1.01 & b_4 & = & 0 & \text{trend chaser} \end{array} \quad (1)$$

$$Rx_t = \sum_{h=1}^{4} n_{h,t}(g_h x_{t-1} + b_j)$$

$$n_{h,t+1} = \exp(\frac{\beta}{a\sigma^2}(g_h x_{t-2} + b_h - Rx_{t-1})(x_t - Rx_{t-1}))/Z_t, \qquad h = 1, 2, 3,$$

Four Belief Types

Rational Route to Randomness:

- $\beta < \beta^*$: fundamental steady state globally stable
- $\beta = \beta^*$: **Hopf bifurcation** of steady state
- $\beta^* < \beta < \beta^{**}$: periodic and quasi-periodic price fluctuations on attracting invariant circle
- high values of β: strange attractors
- $\beta = \infty$: convergence to (locally unstable) fundamental steady state

Theoretical Question:
Is the system close to *homoclinic orbits* and chaos,
when the intensity of choice β is high?

Bifurcation diagram and largest Lyapunov exponent plot

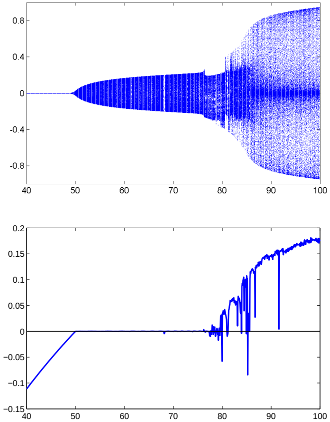


Figure : Bifurcation diagram (top panel) and largest Lyapunov exponent plot (bottom panel) for 4-type model.

Chaotic and noisy chaotic time series, and strange attractor

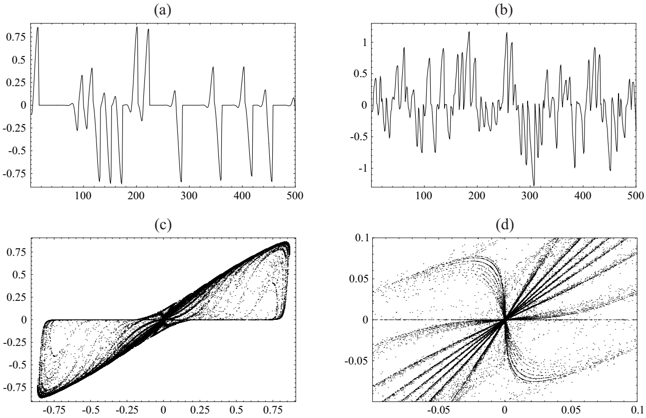


Figure : Chaotic (top left) and noisy chaotic (top right) time series of asset prices in adaptive belief system with four trader types. Strange attractor (bottom left) and enlargement of strange attractor (bottom right).

Forecasting errors

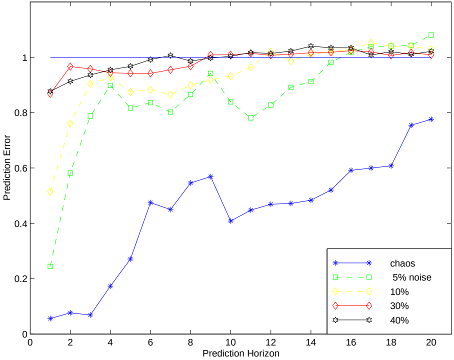
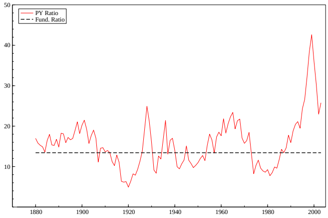
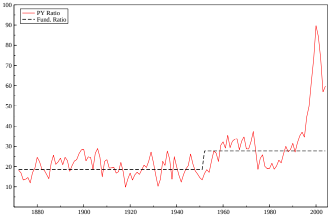


Figure : Forecasting errors for nearest neighbor method applied to chaotic returns series (lowest graph) as well as noisy chaotic returns series, for time horizons $1-20$ and for different noise levels, in ABS with four trader types.

Empirical Validation: PE and PD ratios S&P500, 1871–2003

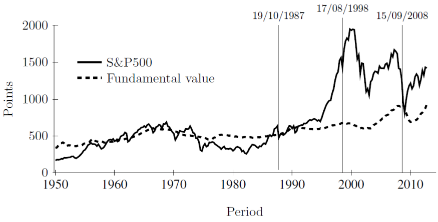


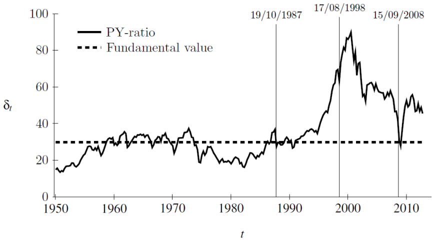
Price-to-Earnings



Price-to-Dividends

S&P 500, 1950-2012 + benchmark fundamental $p_t^* = \frac{1+g}{1+r} y_t$ (g constant growth rate dividends)





BH-Model and risk premium

market clearing (with zero net supply)

$$\sum_{h=1}^{H} n_{h,t} \frac{E_{h,t}[p_{t+1} + y_{t+1}] - (1+r)p_t}{aV_t[R_{t+1}]} = 0$$

equilibrium pricing equation

$$p_t = \frac{1}{1+r} \sum_{h=1}^{H} n_{h,t} E_{h,t}(p_{t+1} + y_{t+1}), \text{ or } r = \sum_{h=1}^{H} n_{h,t} \frac{E_{h,t}[p_{t+1} + y_{t+1} - p_t]}{p_t},$$

estimation:

required rate of return r = risk free interest rate + **risk premium**

Stochastic cash flow with constant growth rate

$\log y_t$ Gaussian random walk with drift:

$$\log y_{t+1} = \mu + \log y_t + \upsilon_{t+1}, \qquad \upsilon_{t+1} \sim \text{i.i.d. } N(0, \sigma_\upsilon^2),$$

This implies

$$\frac{y_{t+1}}{y_t} = e^{\mu + \upsilon_{t+1}} = e^{\mu + \frac{1}{2}\sigma_\upsilon^2} e^{\upsilon_{t+1} - \frac{1}{2}\sigma_\upsilon^2} = (1+g)\varepsilon_{t+1},$$

where $g = e^{\mu + \frac{1}{2}\sigma_\upsilon^2} - 1$ and $\varepsilon_{t+1} = e^{\upsilon_{t+1} - \frac{1}{2}\sigma_\upsilon^2}$, which implies $E_t(\varepsilon_{t+1}) = 1$.
all types **correct beliefs** about cash flows

$$E_{h,t}[y_{t+1}] = E_t[y_{t+1}] = (1+g)y_t E_t[\varepsilon_{t+1}] = (1+g)y_t.$$

RE fundamental benchmark for constant growth cash flow

$$p_t = \frac{1}{1+r} E_t(\mathbf{p}_{t+1} + \mathbf{y}_{t+1})$$

"no bubble" condition implies unique bounded RE **fundamental price** p_t^*:
(discounted sum of expected future dividends)

$$p_t^* = \frac{E_t(y_{t+1})}{1+r} + \frac{E_t(y_{t+2})}{(1+r)^2} + \ldots \quad = \frac{1+g}{1+r} y_t + \frac{(1+g)^2}{(1+r)^2} y_t + \ldots \quad = \frac{1+r}{r-g} y_t.$$

fundamental price to cash flow ratio

$$\delta_t^* = \frac{p_t^*}{y_t} = \frac{1+r}{r-g} = m$$

Reformulation BH-model in terms of price to cash flows

equilibrium pricing equation

$$p_t = \frac{1}{1+r} \sum_{h=1}^{H} n_{h,t} E_{h,t}(p_{t+1} + y_{t+1})$$

in terms of **price-to-cash flows** $\delta_t = p_t / y_t$

$$\delta_t = \frac{1}{R^*} \{1 + \sum_{h=1}^{H} n_{h,t} E_{h,t}[\delta_{t+1}]\}, \qquad R^* = \frac{1+r}{1+g}$$

Heterogeneous Beliefs in terms of price-to-cash flows

deviation price-to-cash flow from fundamental

$$x_t = \delta_t - m = \delta_t - \frac{1+g}{r-g}$$

belief of type h about price-to-cash flow:

$$E_{ht}[\delta_{t+1}] = E_t[\delta_t^*] + f_h(x_{t-1}, ..., x_{t-L}) = m + f_h(x_{t-1}, ..., x_{t-L})$$

pricing equation in **deviations** from fundamental

$$R^* x_t = \sum_{h=1}^{H} n_{ht} f_h(x_{t-1}, ..., x_{t-L}), \qquad R^* = \frac{1+r}{1+g}$$

Evolutionary Fitness Measure

realized net profits in period t

$$U_{ht} = \pi_{ht} = R_t z_{h,t-1} = (p_t + y_t - Rp_{t-1}) \frac{E_{h,t-1}[p_t + y_t - Rp_{t-1}]}{aV_{t-1}[p_t + y_t - Rp_{t-1}]}$$

Assume (in analogy with BH)

$$V_{t-1}[p_t + y_t - Rp_{t-1}] = V_{t-1}[p^*_t + y_t - Rp^*_{t-1}] = y^2_{t-1}\eta^2$$

fitness in deviations from fundamental

$$U_{ht} = \pi_{ht} = \frac{(1+g)^2}{a\eta^2}(x_t - R^* x_{t-1})(E_{h,t-1}[x_t - R^* x_{t-1}])$$

Two-types: Fundamentalists versus trend

Two trader types, with forecasting rules

$$
\begin{aligned}
f_{1t} &= \phi_1 x_{t-1}, \quad 0 \leq \phi_1 < 1 \quad &\textbf{fundamentalists} \\
f_{2t} &= \phi_2 x_{t-1}, \quad \phi_2 > 1, \quad &\text{trend extrapolators}
\end{aligned}
$$

Fractions of the Two Types

fractions of belief types are updated in each period according to discrete choice model (BH 1997,1998)

$$n_{h,t} = \frac{\exp[\beta \pi_{h,t-1}]}{\sum_{k=1}^{H} \exp[\beta \pi_{k,t-1}]} = \frac{1}{1 + \sum_{k \neq h} \exp[-\beta \Delta \pi_{t-1}^{h,k}]},$$

where $\beta > 0$ is **intensity of choice** and
$\Delta \pi_{t-1}^{h,k} = \pi_{h,t-1} - \pi_{k,t-1}$ **difference in realized profits** types h and k
In 2-type case, **fraction of type 1**:

$$n_t = \frac{1}{1 + \exp\left\{-\beta^* \left[(\phi_1 - \phi_2) x_{t-3} (x_{t-1} - R^* x_{t-2})\right]\right\}}$$

Estimation of two type model

in deviations from fundamental; synchronous updating; Boswijk et al., JEDC 2007

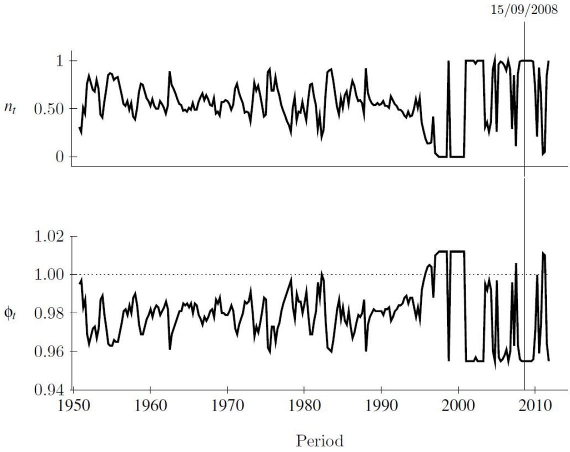
$$R^* x_t = n_t \phi_1 x_{t-1} + (1 - n_t)\phi_2 x_{t-1} + \epsilon_t \qquad R^* = \frac{1+r}{1+g} \approx 1.074$$

- $\phi_1 = 0.762$: **fundamentalists**, mean reversion
- $\phi_2 = 1.135$ **trend extrapolators**
- $\beta \approx 10$

$$\phi_t = \frac{n_t \phi_1 + (1 - n_t)\phi_2}{R^*}$$ **market sentiment**

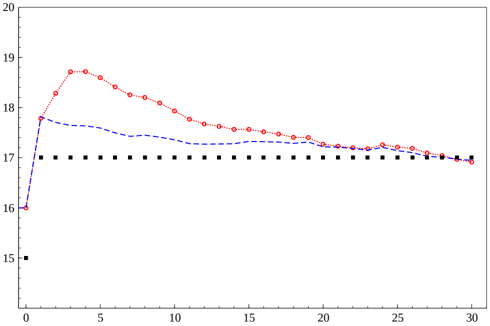
- $\phi_t < 1$: mean reversion;
- $\phi_t > 1$: explosive, trend following

Fraction Fundamentalists & Market Sentiment



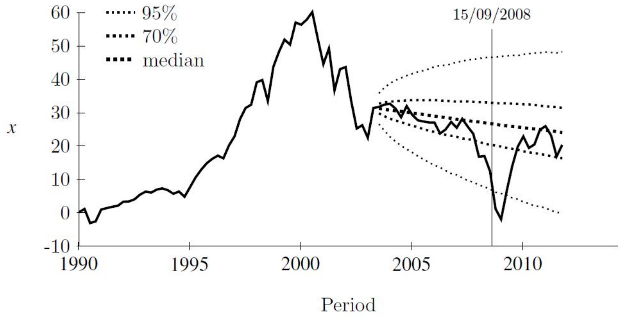
Explanation: dot com bubble triggered by economic fundamentals and **strongly amplified** by trend following behavior

Average Response to Fundamental shock (2000 runs)



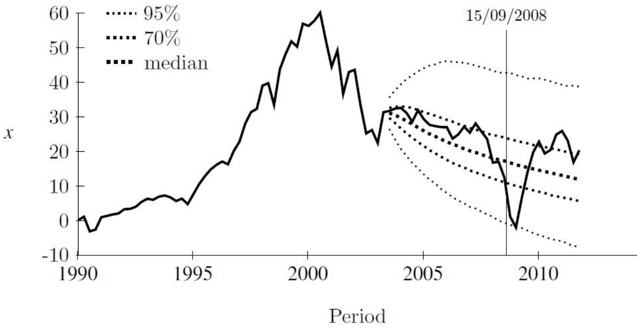
short term **overreaction** and long term **mean reversion**

Financial Crisis Extreme Event in Linear RE Model



Quantiles of 2000 simulated predictions of the PE-ratio
in deviations from fundamental

Financial Crisis not Extreme in Nonlinear Switching Model



Quantiles of 2000 simulated predictions of the PE-ratio in deviations from fundamental

Conclusions

Asset Pricing Model with Heterogeneous Beliefs

- rational route to randomness as β increases, with temporary **bubbles** and **crashes**
    - weak correlation of beliefs: **stable** price behavior;
    - strong coordination of beliefs: **unstable** price dynamics
- counter-examples to Friedman hypothesis:
  fundamentalists do **not** drive out "irrational" **technical analysts**, driven by short run profits
- empirical validation: explanation of bubbles and crashes
- consistent with **learning-to-forecast** laboratory experiments